MULTIFRACTAL MEASURES - FROM SELF-AFFINE TO NONLINEAR

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A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

2021

Identifiers to use to cite or link to this thesis:

DOI: https://doi.org/10.17630/sta/123
http://hdl.handle.net/10023/23786

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This thesis is submitted in partial fulfilment for the degree of
Doctor of Philosophy (PhD)
at the University of St Andrews

March 2021

Acknowledgements

First and foremost I would like to thank my supervisors, Kenneth Falconer and Jonathan Fraser, for all of their support, advice and encouragement over the past four years. Their willingness to help with even the smallest of issues has been phenomenal and has really helped me to make the most of my time in St Andrews. I could not have asked for a better supervisory experience.

On the research side of things I would like to thank my other co-authors, Natalia Jurga, Ian Morris and Han Yu, and EPSRC for funding my PhD. I would also like to thank all the members of the analysis group for creating such a great environment to do research, especially to Mike and Jon for making the the post-seminar trip to the Whey Pat such a highlight of the week.

I would like to thank everyone I have shared "the bunker" (the analysis group PhD office) with over the past four years, especially Doug (for helping me settle in) and Stuart (for all our trips to Brew Co. and Jannetta's). Working from home during the pandemic has definitely not been as fun!

Beyond the realms of maths a very special thanks goes to Mari for our many trips for coffee, lifts home and walks on the beach. I will certainly miss them!

Funding

The work in this thesis was supported by an Engineering and Physical Sciences Research Council (EPSRC) Doctoral Training Grant (EP/N509759/1).

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Abstract

This thesis is based on three papers the author wrote during his time as a PhD student [28, 17, 33].

In Chapter 2 we study L^q-spectra of planar self-affine measures generated by diagonal matrices. We introduce a new technique for constructing and understanding examples based on combinatorial estimates for the exponential growth of certain split binomial sums. Using this approach we find counterexamples to a statement of Falconer and Miao from 2007 and a conjecture of Miao from 2008 concerning a closed form expression for the generalised dimensions of generic self-affine measures.

We also answer a question of Fraser from 2016 in the negative by proving that a certain natural closed form expression does not generally give the L^q-spectrum. As a further application we provide examples of self-affine measures whose L^q-spectra exhibit new types of phase transitions. Finally, we provide new non-trivial closed form bounds for the L^q-spectra, which in certain cases yield sharp results.

In Chapter 3 we study L^q-spectra of measures in the plane generated by certain nonlinear maps. In particular we study attractors of iterated function systems consisting of maps whose components are $C^{1+\alpha}$ and for which the Jacobian is a lower triangular matrix at every point subject to a natural domination condition on the entries. We calculate the L^q-spectrum of Bernoulli measures supported on such sets using an appropriately defined analogue of the singular value function and an appropriate pressure function.

In Chapter 4 we study a more general class of invariant measures supported on the attractors introduced in Chapter 3. These are pushforward quasi-Bernoulli measures, a class which includes the well-known class of Gibbs measures for Hölder continuous potentials. We show these measures are exact dimensional and that their exact dimensions satisfy a Ledrappier-Young formula.

Contents

Chapter 1

Introduction

1.1 What is a fractal?

Fractal geometry is a branch of mathematical analysis that seeks to understand highly irregular geometric objects known as fractals, which we typically think of as subsets of $\mathbb{R}^n$. There is no one strict definition of a fractal but they typically share a variety of common features. These include details at arbitrarily small scales and forms of self-similarity, which is the concept that a fractal is often composed of smaller (possibly distorted) copies of itself.

A classic example is the Sierpiński Triangle, which is displayed in Figure 1.1. Visually it is easy to see the intricate structure of this fractal and it is also readily apparent that it is made up of three smaller copies of itself.

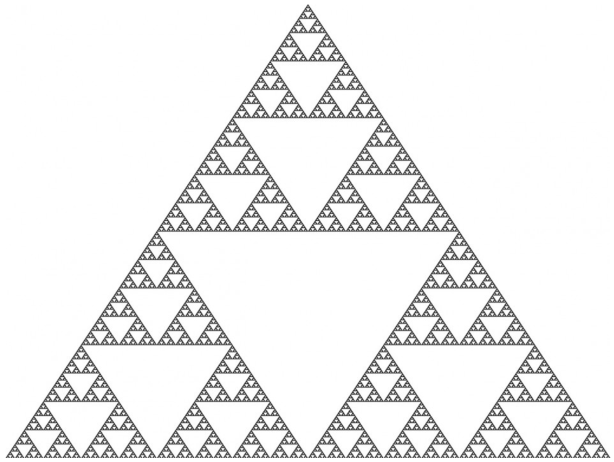

Figure 1.1: The Sierpiński triangle.

Another well-known example of a fractal is the middle third Cantor set. This is generated by removing the open middle third line segment from the unit interval, leaving two disconnected intervals. The middle third of each of these intervals is then removed and the process is continued ad infinitum. The set of points not removed in any iteration is called the middle third Cantor set. This process is displayed visually in Figure 1.2.

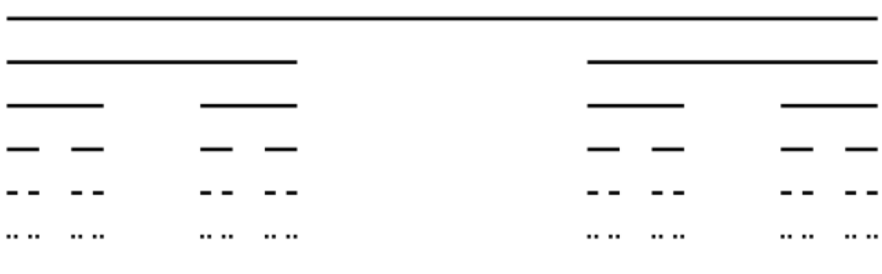

Figure 1.2: Generating the middle third Cantor set.

Fractals can be found throughout mathematics. Notable examples include the graph of Brownian motion in probability theory and the set of badly approximable numbers in number theory. Ideas from fractal geometry can help us to understand these sets better and allow us to quantify their "size" in a highly refined way.

Another reason fractals are of particular interest is because fractal behaviour can be found well beyond the realms of pure mathematics. Examples abound in nature and can be found in diverse settings over a significant range of scales, from the structure of mountain ranges to the branches of a fern.

The term fractal itself was coined by Benoit Mandelbrot in the 1970s, when developments in computer graphics helped him to reveal the beauty of what had previously only been accessible to pure mathematicians to a far wider audience. Despite Mandelbrot's work many of the key concepts that are now central to the field of fractal geometry were in fact introduced far earlier. Felix Hausdorff introduced his eponymous notion of dimension, Hausdorff dimension, in 1918 - the same year that Gaston Julia introduced Julia sets, a class of highly intricate fractal sets which bear his name.

Despite a rich history spanning well over one hundred years fractal geometry still has an important role to play in modern mathematics and has applications well beyond those envisaged by its earliest pioneers. Today fractals can be used to try and understand a diverse range of phenomena, both within pure mathematics and outside of it.

1.2 Constructing fractals

It is natural to ask how one can construct a fractal set. One very common way of doing so is through the use of iterated function systems.

Definition 1.2.1 (Iterated function system (IFS))**.** Let D be a compact subset of $\mathbb{R}^n$, let $\mathcal{I}$ denote a finite index set and write $|\cdot|$ for the Euclidean norm on $\mathbb{R}^n$. Suppose for each $i \in \mathcal{I}$ there is a map $S_i : D \to D$ which is a contraction, that is there exists some $0 < C_i < 1$ such that

$$|S_i(x) - S_i(y)| \leq C_i |x - y|$$

for all $x, y \in D$. Then we call the collection $\{S_i\}_{i\in\mathcal{I}}$ an *iterated function system (IFS).*

The usefulness of IFSs in constructing fractals is illustrated in the following theorem of Hutchinson from 1981, which associates to each a IFS a unique set called the attractor (this result is also often attributed to Moran).

Theorem 1.2.2 (Hutchinson, [31]). *Let $\{S_i\}_{i\in\mathcal{I}}$ be an IFS. Then there exists a unique non-empty compact set F satisfying*

$$F = \bigcup_{i\in\mathcal{I}} S_i(F)$$

We call F the attractor of the IFS.

The attractor is typically a fractal. An example of this is the IFS defined by the maps $S_1, S_2 : [0, 1] \to \mathbb{R}$, where

$$S_1(x) = \frac{x}{3}, \qquad S_2(x) = \frac{x}{3} + \frac{2}{3}. \tag{1.2.3}$$

The attractor of the IFS consisting of S_1 and S_2 can be shown to be the middle third Cantor set (Figure 1.2). In fact the middle third Cantor set is an example of what is known as a self-similar set.

Definition 1.2.4 (Self-similar set). Suppose $\{S_i\}_{i\in\mathcal{I}}$ is an IFS consisting of similarities, i.e. for each $i \in \mathcal{I}$ there exists some $0 < r_i < 1$ such that

$$|S_i(x) - S_i(y)| = r_i|x - y|$$

for all $x, y \in D$. Then we say that the corresponding attractor F is a *self-similar set.*

Famous examples of self-similar sets include both the middle third Cantor set (Figure 1.2) and the Sierpiński triangle (Figure 1.1). Self-similar sets are perhaps the most well understood class of fractals in the literature, as being generated by similarity mappings guarantees that their geometry is particularly regular. A class of fractal sets that is far more challenging to understand is the class of self-affine sets.

Definition 1.2.5 (Self-affine set)**.** Suppose $\{S_i\}_{i\in\mathcal{I}}$ is an IFS consisting of affine maps, i.e. for each $i \in \mathcal{I}$ there exists some non-singular linear transformation A_i on $\mathbb{R}^n$ and $t_i \in \mathbb{R}^n$ such that

$$S_i(x) = A_i(x) + t_i$$

for all $x \in D$. Then we say that the corresponding attractor F is a *self-affine set*.

The reason self-affine sets are more difficult to work with lies in the fact that the defining contractions may contract by different amounts in different directions. This does however mean they have a richer geometry than self-similar sets and often resemble natural phenomena more closely. An example of this can be seen in Figure 1.3, which displays three self-affine sets introduced by Fraser in [24], examples of what he called *box-like self-affine sets*.

Figure 1.3: Three of Fraser's box-like self-affine sets, image taken from [25].

Beyond self-affine sets one can study fractals that are generated by nonlinear maps. The simplest example of such sets are self-conformal sets.

Definition 1.2.6 (Self-conformal set)**.** Suppose $\{S_i\}_{i\in\mathcal{I}}$ is an IFS where for each $i \in \mathcal{I}$ the map $S_i : D \to D$ extends to an injective contraction on some open set $U \supset D$ such that the map $S_i : U \to U$ is C^1 and the derivative $S_i'(x) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|S_i'(x)y| = |S_i'(x)||y| \neq 0$$

for all $x \in U$ and $y \in \mathbb{R}^n \setminus \{0\}$. Then we say that the corresponding attractor F is a *self-conformal set*.

Self-conformal sets can be thought of as a nonlinear analogue of self-similar sets. Indeed another way of thinking of self-conformal sets is that for each map S_i in the defining IFS the derivative S_i' is a similarity for each $x \in U$.

1.3 Dimension theory

So far we have looked at several different classes of fractal set and seen how they can be constructed. Moving beyond this we shall now consider how the size of a fractal set can be measured.

Due to the intricate structure of fractals at small scales this can often be a challenging problem and many issues can arise. Take for instance the Sierpiński triangle (Figure 1.1). Although we think of it as a subset of $\mathbb{R}^2$ it can be shown that the Sierpiński triangle has zero area, but infinite length (more formally its one-dimensional outer Lebesgue measure is infinite). It therefore appears that our traditional way of measuring subsets of $\mathbb{R}^n$ (namely Lebesgue measure) is inadequate for fractal sets.

The issue here is one of dimension - the Sierpiński triangle simply does not fit nicely into our traditional notion of integer dimensions. To try and rectify this mathematicians have introduced a variety of different notion of dimension that can take on non-integer values. It turns out these are ideally suited for understanding the complicated geometry of fractal sets.

One of the simplest and most useful of these dimensions is box dimension.

Definition 1.3.1 (Box dimension)**.** Let $X \subset \mathbb{R}^n$ be bounded and non-empty and let $N_\delta(X)$ denote the minimal number of sets of diameter at most δ needed to cover X. The *upper and lower box dimensions* of X are defined to be

$$\overline{\dim}_B X = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta} \quad \text{and} \quad \underline{\dim}_B X = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}$$

respectively. If $\overline{\dim}_B X$ and $\underline{\dim}_B X$ coincide then we define the *box dimension* of X, denoted $\dim_B X$, to be their common value.

For a self-similar set F generated by an IFS which satisfies the open set condition (that is there exists some non-empty open set V such that $\cup_{i \in \mathcal{I}} S_i(V) \subset V$ with the union disjoint) the box dimension of F is given by a closed form expression, namely it is given by the unique s such that

$$\sum_{i \in \mathcal{I}} r_i^s = 1, \tag{1.3.2}$$

where r_i denotes the similarity ratio of S_i. It is easy to see from (1.2.3) that the contraction ratios of the middle third Cantor set are $r_1 = r_2 = 1/3$, so (1.3.2) tells us that its box dimension is $\log 2 / \log 3 \approx 0.63$. The box dimension of the Sierpiński triangle can also be calculated in this way, (1.3.2) gives it to be $\log 3 / \log 2 \approx 1.58$.

One can equivalently use a variety of different approaches to cover a set X to calculate box dimension, for instance by using a δ-mesh and calculating the number of cubes which intersect X. This is displayed visually with the Sierpiński triangle in Figure 1.4.

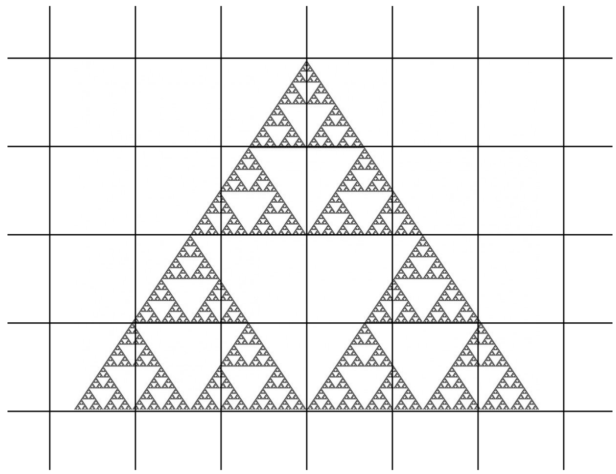

Figure 1.4: Covering the Sierpiński triangle with a δ-mesh.

Another dimension which as been extensively studied is Hausdorff dimension.

Definition 1.3.3 (Hausdorff outer measure and dimension)**.** Let $X \subseteq \mathbb{R}^n$ and $s \geq 0$. For $\delta > 0$ we let

$$\mathcal{H}^s_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i \in \mathbb{N}} \text{ is a } \delta\text{-cover of } X \right\}.$$

We then define the s-dimensional *Hausdorff outer measure* of X to be

$$\mathcal{H}^s(X) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(X)$$

and we define the *Hausdorff dimension* of X by

$$\dim_H(X) = \inf \{s : \mathcal{H}^s(X) = 0\} = \sup \{s : \mathcal{H}^s(X) = \infty\}.$$

Hausdorff dimension is in some ways a more "well-behaved" notion of dimension than box dimension. In particular Hausdorff dimension is stable under countable unions, which means that if $\{F_i\}_{i\in\mathcal{I}}$ is a countable collection of sets then $\dim_H\left(\bigcup_{i\in\mathcal{I}}\{F_i\}\right) = \sup_{i\in\mathcal{I}}\{\dim_H F_i\}$. This is not the case with box dimension and this means that one can construct countable sets with positive box dimension - something impossible with Hausdorff dimension. Hausdorff outer measure is also of interest beyond its use in calculating the Hausdorff dimension, indeed when restricted to an appropriate class of measurable sets it becomes a measure known as *Hausdorff measure*, Hausdorff measure can be viewed as a refinement of n-dimensional Lebesgue measure (indeed they are comparable when $s = n$).

In terms of the relationship between the two notions of dimension for any bounded set $X \subset \mathbb{R}^n$ we always have $\dim_H X \leq \overline{\dim}_B X$. There are many settings where Hausdorff dimension and box dimension coincide, for instance if X is a self-similar or self-conformal set which satisfies the open set condition. Indeed if if X is a self-similar set which satisfying the open set condition then its Hausdorff dimension is also given by (1.3.2).

In the self-affine case however equality is far from guaranteed, for instance in the classic example of Bedford-McMullen carpets Bedford [7] and McMullen [41] independently showed that the Hausdorff dimension of these carpets is in general strictly less than the box dimension.

Despite the intricate geometry of self-affine sets making them more challenging to work with than self-similar sets, there has been significant progress in understanding their dimension theory. Notably Falconer, in his seminal 1988 paper [11], introduced a highly successful formula for calculating the Hausdorff dimension of self-affine sets. To begin with recall that for an $n \times n$ matrix A the singular values of A are the positive square roots of the eigenvalues of $A^T A$. Given a self-affine set F generated by the IFS $\{S_i\}_{i\in\mathcal{I}}$, we write $\mathcal{I}^k$ for the set of all k-length sequences over $\mathcal{I}$ and we write $\mathcal{I}^* = \bigcup_{k\geq 1} \mathcal{I}^k$ for the set of all finite sequences over $\mathcal{I}$. For $\mathtt{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ we let $S_\mathtt{i} = S_{i_1} \circ \cdots \circ S_{i_k}$ and write $\alpha_1(\mathtt{i}) \geq \cdots \geq \alpha_n(\mathtt{i})$ for the singular values of the linear part of $S_\mathtt{i}$. We can now introduce the singular value function.

Definition 1.3.4 (Singular value function)**.** Let $\{S_i\}_{i\in\mathcal{I}}$ denote an affine IFS. For $0 \leq s \leq n$ we define the *singular value function* $\phi^s : \mathcal{I}^* \to (0, \infty)$ by

$$\phi^s(\mathtt{i}) = \alpha_1(\mathtt{i}) \cdots \alpha_{\lceil s\rceil -1}(\mathtt{i})\alpha_{\lceil s\rceil}(\mathtt{i})^{s-\lceil s\rceil+1}.$$

For $s \geq n$ we define $\phi^s(\mathtt{i})$ to be

$$\phi^s(\mathtt{i}) = (\alpha_1(\mathtt{i}) \cdots \alpha_n(\mathtt{i}))^{s/n}.$$

We shall occasionally write $\phi^s(S_\mathtt{i})$ for $\phi^s(\mathtt{i})$ when we wish to emphasise the underlying IFS.

The singular value function contains a lot of useful information about the geometry of F which is especially helpful in understanding its Hausdorff dimension. It can be shown that there exists a unique s which satisfies

$$\lim_{k\to\infty} \left(\sum_{\mathtt{i}\in\mathcal{I}^k} \phi^s(\mathtt{i}) \right)^{1/k} = 1.$$

We write $d(S_i : i \in \mathcal{I})$ to denote this s and refer to $d(S_i : i \in \mathcal{I})$ as the *singularity dimension* of $\{S_i\}_{i\in\mathcal{I}}$.

The importance of the singularity dimension is given by the following two theorems. The first of these theorems states that for a self-affine set $d(S_i : i \in \mathcal{I})$ is always an upper bound for the upper box dimension.

Theorem 1.3.5 (Falconer, [11]). *Let $\{S_i\}_{i\in\mathcal{I}}$ denote an affine IFS on $\mathbb{R}^n$ with corresponding attractor F. Then*

$$\overline{\dim}_B F \leq d(S_i : i \in \mathcal{I}).$$

The second theorem tells us that for "almost all" self-affine sets $d(S_i : i \in \mathcal{I})$ gives us both the Hausdorff and the box dimension. We write $m = |\mathcal{I}|$ and $\mathcal{L}^{mn}$ for mn-dimensional Lebesgue measure.

Theorem 1.3.6 (Falconer, [11]). *Let $\{S_i\}_{i\in\mathcal{I}}$ denote a collection of contracting linear maps on $\mathbb{R}^n$ with contraction ratio strictly less than $1/3$ and let $t_1, \ldots, t_m \in R^n$. Then the unique non-empty compact set F guaranteed by Theorem 1.2.2 for which*

$$F = \bigcup_{i\in I} (S_i(F) + t_i)$$

satisfies

$$\dim_B F = \dim_H F = \min\{n, d(S_i : i \in \mathcal{I})\}$$

for $\mathcal{L}^{mn}$ almost all for $(t_1, \ldots, t_m) \in \mathbb{R}^{mn}$.

This result tells us that generically a self-affine set F has equal box and Hausdorff dimensions, with both of these quantities given by $\min\{n, d(S_i : i \in \mathcal{I})\}$. The question of precisely when these two dimensions coincide has a attracted significant interest in recent years, see for instance [2, 29]. Finally it should be noted that the contraction ratios in Theorem 1.3.6 can in fact be relaxed from 1/3 to 1/2. This was shown by Solomyak [48] who also showed that 1/2 is optimal.

1.4 Multifractals

Up to this point we have been concerned with fractal *sets*, but one can also study fractal *measures*, which are more commonly known as multifractals. As with fractal sets there is no one strict definition of a multifractal but they instead have a variety of common features, most of which are shared with the set case (e.g. details at arbitrarily small scales, forms of self-similarity etc.) For the purpose of this thesis the measures μ we shall study are all compactly supported Borel probability measures.

A common way to construct multifractals is to start with a fractal set and then to construct a measure supported on it in a natural way. This is how self-similar, self-affine and self-conformal measures are constructed.

Definition 1.4.1 (Self-similar, self-affine and self-conformal measures)**.** Let F be self-similar (respectively self-affine, self-conformal) set given by the IFS $\{S_i\}_{i\in\mathcal{I}}$, and let $\{p_i\}_{i\in\mathcal{I}}$ be a probability vector with each $p_i \in (0,1)$. Then there is a unique Borel probability measure μ on $\mathbb{R}^n$ satisfying

$$\mu = \sum_{i\in\mathcal{I}} p_i \ \mu \circ S_i^{-1},$$

which we call the *self-similar* (respectively *self-affine, self-conformal*) *measure* associated to $\{S_i\}_{i\in\mathcal{I}}$ and $\{p_i\}_{i\in\mathcal{I}}$.

The construction of a self-similar measure on the middle third Cantor set is illustrated in Figure 1.5.

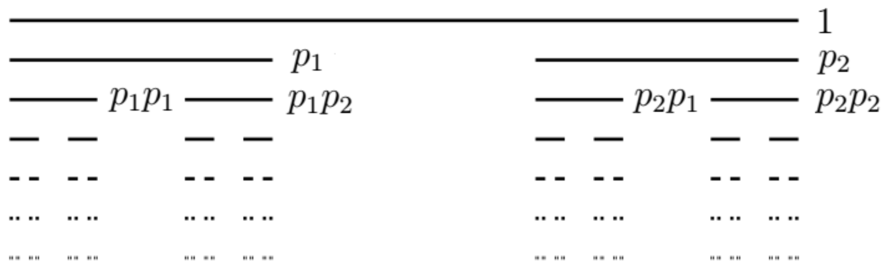

Figure 1.5: Constructing a self-similar measure on the middle third Cantor set using the probability vector (p_1, p_2). The unit interval is given mass 1, which is then repeatedly split in the ratio $p_1 : p_2$ on the intervals in the construction.

1.5 Dimension of measures

Just as we seek to understand the dimension of fractal sets, we are are also interested in the dimension theory of multifractals. We begin by looking at the Hausdorff dimension of a measure.

Definition 1.5.1 (Hausdorff dimension of a measure)**.** Let μ denote a Borel probability measure. We define the *Hausdorff dimension* of μ to be

$$\dim_H \mu = \inf\{\dim_H X : X \text{ is a Borel set with } \mu(X) = 1\}.$$

The Hausdorff dimension of a measure gives us global information about the measure. One may also be interested in more local properties of a measure and for that we can look at local dimension.

Definition 1.5.2 (Local dimension and exact dimensionality)**.** Let $B(x, r)$ denote the closed ball of radius r centred at x. We define the *upper and lower local dimensions* of

μ at x by

$$\overline{\dim}_{\mathrm{loc}}(x) = \limsup_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log r} \quad \text{and} \quad \underline{\dim}_{\mathrm{loc}}(x) = \liminf_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log r}$$

respectively. If $\overline{\dim}_{\mathrm{loc}}(x)$ and $\underline{\dim}_{\mathrm{loc}}(x)$ coincide then we define the *local dimension* of μ at x, denoted $\dim_{\mathrm{loc}}(x)$, to be their common value.

If $\dim_{\mathrm{loc}}(x)$ exists and is constant for μ almost all x then we say that μ is *exact dimensional*. We write $\dim \mu$ for this almost sure value, which we term the *exact dimension* of μ.

If a measure μ is exact dimensional then $\dim \mu = \dim_H \mu$, see for instance [19]. Furthermore it is worth mentioning that not all measures are exact dimensional. For instance if one considers a collection of ergodic measures (see Section 1.7) which are each exact dimensional but have different exact dimensions, then convex sums of these measures will fail to be exact dimensional. A perhaps more interesting example can be found in [10, Theorem 17.5.13], where it is shown that certain Patterson-Sullivan measures (a class of measures studied in hyperbolic geometry) are not exact dimensional.

In Chapter 4 we will show that a class of measures supported on sets generated by an IFS consisting of nonlinear maps are exact dimensional (Theorem 4.2.5).

1.6 The multifractal formalism

Of central importance in the study of multifractals are the level sets of the local dimension. This is because when taken together these sets tell us about the behaviour of the measure in question at all points in its support. This motivates the definition of the fine multifractal spectrum.

Definition 1.6.1 (Fine multifractal spectrum)**.** For a compactly supported Borel probability measure μ and $\alpha \geq 0$ let

$$F_\alpha = \{x \in \mathbb{R}^n : \dim_{\mathrm{loc}}(x) = \alpha\}.$$

We then define the *fine multifractal spectrum* of μ to be the function

$$f_H(\alpha) = \dim_H F_\alpha.$$

Calculating the fine multifractal spectrum of a measure is a problem of central importance in the theory of multifractals. It is however an incredibly challenging problem and one where progress is still elusive. Beyond self-similar and self-conformal measures very little is known, with one of the few exceptions being self-affine measures on Bedford-McMullen carpets. King [34] gave an explicit formula for the fine multifractal spectrum of these measures under a very strong separation condition and Olsen [45] was able to generalise his result to higher dimensions. Jordan and Rams [32] were also able generalise King's result by removing the very strong separation condition.

One common approach to calculating the fine multifractal spectrum is to study whether the *multifractal formalism* holds. The multifractal formalism states that the fine multifractal spectrum can be found by calculating the Legendre transform of a quantity known as the L^q-spectrum, which will be a key focus of this thesis.

The L^q-spectrum is defined in terms of moment sums of a compactly supported Borel probability measure μ. Let $\delta > 0$ and let $\mathcal{D}_\delta$ denote the set of closed cubes in the δ-mesh on $\mathbb{R}^n$ that have positive μ-measure. Write

$$\mathcal{D}_\delta^q(\mu) = \sum_{Q \in \mathcal{D}_\delta} \mu(Q)^q. \tag{1.6.2}$$

We can now define the L^q-spectrum.

Definition 1.6.3 (L^q-spectrum)**.** If μ is a compactly supported Borel probability measure on $\mathbb{R}^n$ then for $q \geq 0$ the *upper and lower L^q-spectra* of μ are defined to be

$$\overline{\tau}_\mu(q) = \overline{\lim}_{\delta \to 0} \frac{\log \mathcal{D}_\delta^q(\mu)}{-\log \delta} \text{ and } \underline{\tau}_\mu(q) = \underline{\lim}_{\delta \to 0} \frac{\log \mathcal{D}_\delta^q(\mu)}{-\log \delta} \tag{1.6.4}$$

respectively. If these values coincide then we define the *L^q-spectrum* of μ, denoted by $\tau_\mu(q)$, to be their common value.

The L^q-spectrum has several useful properties, perhaps the most notable being its aforementioned connection to the fine multifractal spectrum. The Legendre transform of τ_μ is given by

$$f(\alpha) = \inf_{q \geq 0} \{\tau_\mu(q) + \alpha q\}$$

(it is also possible to define τ_μ and hence $f(\alpha)$ for negative q, although we do not pursue this). The Legendre transform of the L^q-spectrum is always an upper bound for the fine multifractal spectrum, i.e.

$$f_H(\alpha) \leq f(\alpha), \tag{1.6.5}$$

see for instance [16, Proposition 17.2, Lemma 17.3]. In many cases there is equality in (1.6.5), in which case we say that the multifractal formalism holds.

Beyond its importance in the multifractal formalism the L^q-spectrum also has a number of other interesting properties. If μ is a compactly supported Borel measure on $\mathbb{R}^n$, $q, r \geq 0$, $t \in [0, 1]$ and $\delta > 0$ then

$$\mathcal{D}_\delta^{tq+(1-t)r}(\mu) = \sum_{Q \in \mathcal{D}_\delta} \mu(Q)^{tq+(1-t)r}$$

$$\leq \left(\sum_{Q \in \mathcal{D}_\delta} \mu(Q)^q \right)^t \left(\sum_{Q \in \mathcal{D}_\delta} \mu(Q)^r \right)^{1-t}$$

$$= \mathcal{D}_\delta^q(\mu)^t \mathcal{D}_\delta^r(\mu)^{1-t} \tag{1.6.6}$$

where the inequality is given by Hölder's inequality. It is a consequence of (1.6.6) that the L^q-spectrum is always a convex function (and therefore continuous on $(0, \infty)$). Furthermore it can easily be seen from the definition that it is a decreasing function and $\tau_\mu(1)$ is always equal to 0. The L^q-spectrum also has a close relationship to various other notions of dimension for both sets and measures, one of which is box dimension. Writing $\mathrm{supp}(\mu)$ to denote the support of μ, it is easy to see from (1.6.2) that if $q = 0$ then $\mathcal{D}_\delta^0(\mu) = N_\delta(\mathrm{supp}(\mu))$. This implies that the upper and lower box dimensions of $\mathrm{supp}(\mu)$ are given by $\overline{\tau}_\mu(0)$ and $\underline{\tau}_\mu(0)$ respectively. If these coincide then

$$\dim_B \mathrm{supp}(\mu) = \tau_\mu(0). \tag{1.6.7}$$

The L^q-spectrum is also related to the Hausdorff dimension of the measure μ. Ngai [43] showed that if the L^q-spectrum is differentiable at $q = 1$ then

$$\dim_H \mu = -\tau_\mu'(1).$$

It is worth mentioning that for $q \geq 0$ the L^q-spectrum of Lebesgue measure on $[0, 1]$ is $1 - q$. This has motivated some authors to instead study the L^q-spectrum normalised by $1-q$, a giving a non-increasing function of q known as the generalised q-dimensions.

Definition 1.6.8 (Generalised q-dimensions)**.** If μ is a compactly supported Borel probability measure on $\mathbb{R}^n$ then for $q \geq 0, q \neq 1$ the *generalised q-dimensions* of μ are defined to be

$$D_q(\mu) = \frac{\tau_\mu(q)}{1-q},$$

provided the appropriate limit in the definition of the L^q-spectrum exists.

Whilst more work has been done on calculating the L^q-spectrum than has been done on calculating the fine multifractal spectrum it can still present a major challenge in many situations. One area that is well understood though is the L^q-spectrum of self-similar measures satisfying the open set condition. To do this we introduce a function $\beta : \mathbb{R} \to \mathbb{R}$, where β is defined by

$$\sum_{i \in \mathcal{I}} p_i^q r_i^{\beta(q)} = 1. \tag{1.6.9}$$

For $q \geq 0$ the L^q-spectrum is then given by

$$\tau_\mu(q) = \beta(q)$$

The expression (1.6.9) can be viewed as an analogue of (1.3.2) for measures. Indeed (1.3.2) can be obtained from (1.3.2) by using (1.6.7).

The expression for τ_μ given in (1.6.9) is particularly nice as it is a *closed form expression*. Closed form expressions for L^q-spectra of self-affine measures are addressed in Chapter 2 of this thesis. We show that the expected candidate for τ_μ in this setting (an analogue of β) does not always give the L^q-spectrum (Theorem 2.3.14).

Moving beyond self-affine measures presents considerable difficulties. In Chapter 3 we calculate the L^q-spectrum of a class of measures supported on IFSs consisting of nonlinear mappings (Theorem 3.2.3 and Theorem 3.4.24), although as is to be expected in this far more complicated situation we do not obtain a closed form expression.

1.7 Symbolic dynamics

Symbolic dynamics seeks to understand the dynamical properties of sequence spaces under the shift map. Whilst sequence spaces are of interest in their own right, symbolic dynamics can also be used to study other dynamical systems by representing them symbolically. It will be a particularly useful concept for us as it can be used to define measures on fractal sets and it can also be used in dimension calculations.

We wish to study an IFS $\{S_i\}_{i \in \mathcal{I}}$ with attractor F. We begin by setting up some notation for sequences. For $n \in \mathbb{N}$ we write $\mathcal{I}^n$ to denote the set of all sequences of length n over $\mathcal{I}$ and for $\mathtt{i} = (i_1, i_2, \dots, i_n) \in \mathcal{I}^n$ we write $S_{\mathtt{i}} = S_{i_1} \circ \cdots \circ S_{i_n}$.

We write $\mathcal{I}^* = \bigcup_{n \geq 1} \mathcal{I}^n$ for the set of all finite sequences over $\mathcal{I}$. For finite sequences $\mathtt{i}, \mathtt{j} \in \mathcal{I}^*$ we write $|\mathtt{i}|$ to denote the length of $\mathtt{i}$ and $\mathtt{ij}$ to denote the concatenation of $\mathtt{i}$ and $\mathtt{j}$.

We write $\Sigma = \mathcal{I}^{\mathbb{N}}$ for the set of infinite sequences over $\mathcal{I}$. For an infinite sequence $\mathtt{i} = (i_1, i_2, \dots) \in \Sigma$ and $n \in \mathbb{N}$ we let $\mathtt{i}|n = (i_1, i_2, \dots, i_n) \in \mathcal{I}^n$ denote the restriction of $\mathtt{i}$ to its first n symbols. For $\mathtt{i}, \mathtt{j} \in \Sigma$ we write $\mathtt{i} \wedge \mathtt{j} \in \mathcal{I}^* \cup \Sigma$ for the longest prefix which is common to both $\mathtt{i}$ and $\mathtt{j}$. This allows us to define a metric d on Σ by letting $d(\mathtt{i}, \mathtt{j}) = 2^{-|\mathtt{i} \wedge \mathtt{j}|}$ if $\mathtt{i} \neq \mathtt{j}$ and $d(\mathtt{i}, \mathtt{j}) = 0$ if $\mathtt{i} = \mathtt{j}$.

Of particular importance in symbolic dynamics is the notion of a cylinder set.

Definition 1.7.1 (Cylinder set)**.** For $\mathtt{i} \in \mathcal{I}^*$ we define $[\mathtt{i}] \subseteq \Sigma$ to be the set of all $\mathtt{l} \in \Sigma$ of the form $\mathtt{l} = \mathtt{ij}$ for some $\mathtt{j} \in \Sigma$. We call $[\mathtt{i}]$ the *cylinder set* corresponding to $\mathtt{i}$.

The cylinder set $[\mathtt{i}]$ can be thought of as the set of all infinite sequences which begin with the finite sequence $\mathtt{i}$. Assuming $\mathcal{I}$ is equipped with the discrete topology then the topology induced on Σ by the metric d is the product topology. It can be shown that the Borel σ-algebra corresponding to this topology is generated by the set of all finite unions of cylinder sets.

A useful aspect of symbolic spaces is that one can define measures on them which one can then transfer to the attractor F. This is done by via the natural projection map $\Pi : \Sigma \to \mathbb{R}^n$ given by $\Pi(i_1, i_2 \dots) = \lim_{n\to\infty} S_{i_1} \circ \cdots \circ S_{i_n}(x)$, where $x \in \mathbb{R}^n$ is some arbitrary point. Then if one has a Borel probability measure m on Σ the pushforward measure $\mu = m \circ \Pi^{-1}$ is a probability measure on F.

We write $\sigma : \Sigma \to \Sigma$ for the left shift map, i.e. the map for which $\sigma((i_1, i_2, \dots)) = (i_2, i_3, \dots)$. We say that a Borel measure μ supported on F is *invariant* if there exists a σ-invariant Borel measure m on Σ with $\mu = m \circ \Pi^{-1}$, that is $m(\sigma^{-1}B) = m(B)$ for all Borel sets $B \subseteq \Sigma$. Similarly we say that a Borel measure μ supported on F is ergodic if there exists an ergodic Borel measure m on Σ with $\mu = m \circ \Pi^{-1}$, i.e. m is a σ-invariant measure and if $B \subseteq \Sigma$ is a Borel set with $\sigma^{-1}B = B$, then $m(B) = 0$ or 1.

The first class of measures on Σ we shall consider are Bernoulli measures.

Definition 1.7.2 (Bernoulli measure)**.** We say that a measure m on Σ is a *Bernoulli measure* if there exists a probability vector $\{p_i\}_{i\in\mathcal{I}}$ such that

$$m([i_1, \dots, i_n]) = p_{i_1} \cdots p_{i_n}$$

for all finite sequences $(i_1, i_2, \dots, i_n) \in \mathcal{I}^*$.

We have already seen several examples of Bernoulli measures, in particular self-similar, self-affine and self-conformal measures (Definition 1.4.1) are all pushforward Bernoulli measures.

Given a function $f : \Sigma \to \mathbb{R}$ we write $f^n(\mathbf{i}) = \sum_{k=0}^{n-1} f(\sigma^k \mathbf{i})$. This allows us to introduce

Gibbs measures.

Definition 1.7.3 (Gibbs measure)**.** We say that a measure m on Σ is a *Gibbs measure* with potential f if there exist constants $K > 0$ and $P \in \mathbb{R}$ and a continuous function $f : \Sigma \to \mathbb{R}$ such that

$$K^{-1} \leq \frac{m([i_1, \ldots, i_n])}{e^{f^n(\mathtt{i}) - nP}} \leq K$$

for all $\mathtt{i} \in \Sigma, n \in \mathbb{N}$.

The final class of measures we shall look at are quasi-Bernoulli measures.

Definition 1.7.4 (Quasi-Bernoulli measure)**.** We say that a measure m on Σ is *quasi-Bernoulli* if there exists some $L > 0$ such that for all $\mathtt{i}, \mathtt{j} \in \mathcal{I}^*$

$$L^{-1} m([\mathtt{i}]) m([\mathtt{j}]) \leq m([\mathtt{ij}]) \leq L m([\mathtt{i}]) m([\mathtt{j}]). \tag{1.7.5}$$

Any Bernoulli measure is a Gibbs measure, and Gibbs measures for Hölder continuous potentials are quasi-Bernoulli. Furthermore it was shown by Bárány, Käenmäki and Morris [4] that the set of quasi-Bernoulli measures strictly includes the set of Gibbs measures.

We have now seen several classes of measure which can be defined on the symbolic space which one can pushforward onto the attractor. In order to calculate the dimension of these measures there are two further concepts from symbolic dynamics which can prove useful. The first such concept is entropy, which can defined using the Shannon-McMillian-Breiman theorem.

Theorem 1.7.6 (Shannon-McMillan-Breiman, [47, 40, 8])**.** *Let m be an ergodic Borel* *probability measure on* Σ *and let* $\mu = m \circ \Pi^{-1}$. *There exists a constant* $h(\mu) \geq 0$ *such that for* m*-almost all* $\mathtt{i} \in \Sigma$,

$$h(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log m([\mathtt{i}|n]). \tag{1.7.7}$$

We call $h(\mu)$ *the* entropy of μ.

Our second useful concept is that of Lyapunov exponents, which for convenience we introduce only in the self-affine setting and only in $\mathbb{R}^2$. In order to do this we require Kingman's subadditive ergodic theorem.

Theorem 1.7.8 (Kingman, [35])**.** *Let* m *be an ergodic Borel probability on* Σ *and let* $\{g_k\}_{k\in\mathbb{N}}$ *be a sequence of* L^1 *functions on* Σ. *If* $g_{k+l}(\mathtt{i}) \leq g_k(\mathtt{i}) + g_l(\sigma^k \mathtt{i})$ *for all* $\mathtt{i} \in \Sigma$ *and* $k, l \in \mathbb{N}$, *then* $\lim_{k\to\infty} g_k(\mathtt{i})/k$ *is constant for* m *almost all* $\mathtt{i} \in \Sigma$.

As the affine IFS we will consider is in the plane this implies that the singular values α_1 and α_2 are sub and super-multiplicative respectively. Kingman's subadditive ergodic theorem then allows us to define Lyapunov exponents.

Theorem 1.7.9 (Lyapunov exponents (self-affine setting))**.** *Let* m *be an ergodic Borel probability measure on* Σ, *let* $\mu = m \circ \Pi^{-1}$ *and let* $\{S_i\}_{i\in\mathcal{I}}$ *be an affine IFS on* $\mathbb{R}^2$ *with attractor* F. *Then there exist constants* $0 < \chi_1(\mu) \leq \chi_2(\mu)$ *such that for* m*-almost all* $\mathtt{i} \in \Sigma$,

$$\chi_1(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log \alpha_1 \left(S_{\mathtt{i}|n}\right)$$

and

$$\chi_2(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log \alpha_2 \left(S_{\mathtt{i}|n}\right).$$

We call the constants $\chi_1(\mu), \chi_2(\mu)$ *the* Lyapunov exponents of μ.

Entropy and Lyapunov exponents are important as in many cases if the measure μ is exact dimensional (Definition 1.5.2) then its exact dimension can be given by an expression involving both of these quantities and the dimension of an appropriate projected measure. In the non-conformal setting such expressions are often known as Ledrappier-Young formulae. In Chapter 4 we calculate Ledrappier-Young formulae for a class of planar non-conformal measures (Theorem 4.2.5).

1.8 Some notation

We conclude the introduction by introducing some notation which will be used throughout the thesis, usually when manipulating sequences and functions.

If $(a_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$ are sequences such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

we say that $(a_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ are *asymptotically equivalent* and write $\backsim$ to denote this.

For $x, y \in \mathbb{R}^+$ we write $x \lesssim y$ to mean that $x \leq Cy$ for some absolute constant $C > 0$. If we wish to emphasize that this constant depends on some other parameter, θ say, we write $x \lesssim_\theta y$. If both $x \lesssim y$ and $y \lesssim x$ we write $x \asymp y$. In this case we say that x and y are *comparable*.

Chapter 2

Closed form expressions for L^q-spectra

2.1 Background

In the first half of this chapter we study closed form expressions for the L^q-spectrum of self-affine measures in the plane. In Chapter 1 we have seen that the L^q-spectrum is a key concept in the study of multifractals, both in terms of the information it contains and its importance in the multifractal formalism.

Whilst for self-similar measures satisfying the open set condition the L^q-spectrum can be calculated using a straightforward closed form expression (1.6.9), for self-affine measures it can be very difficult to compute. Indeed the L^q-spectrum has only been calculated in some limited specific cases, see for instance [23, 25], although there are some examples where a generic formula has been found [5, 14, 15].

In these small number of cases where a formula has been found for the L^q-spectrum it is in general not given by a closed form expression, which makes it both challenging to calculate explicitly and makes further theoretical information hard to glean. Despite this some work has looked at closed form expressions, see [18, 25, 42].

We begin by looking at the setting of Fraser [25] and Feng-Wang [23], where the self-affine measures are in the plane and generated by diagonal systems i.e. IFSs consisting of affine maps whose linear parts can be represented by diagonal matrices.

Fraser [25, Theorem 2.10] was able to provide closed form expressions for the L^q-spectra of these diagonal systems in many cases, however he often required extra assumptions on the defining IFS and probability vector. He asked if it was possible for these additional technical assumptions to be removed and if his formula held more generally [25, Question 2.14].

We show that the answer to Fraser's question is negative, in particular we provide a simple, explicit family of counterexamples, see Theorem 2.3.14. Despite the fact that the expected closed form expression does not hold we are able to obtain some new, non-trivial, closed form bounds for the L^q-spectra, see Theorem 2.3.24. We obtain examples of self-affine measures with L^q-spectra that display new types of phase transitions, see Theorem 2.3.22. In particular we are able to construct examples where the L^q-spectrum is differentiable at $q = 1$ but not analytic in any neighbourhood of $q = 1$.

In the second half of this chapter we turn to the setting of Falconer-Miao [18] and Miao [42] where the self-affine measures are generated by upper triangular matrices in $\mathbb{R}^n$. The paper [18] was mainly concerned with dimensions of self-affine sets, but towards the end it states a closed form expression for the generalised q-dimensions (Definition 1.6.8) in a natural generic setting [18, Theorem 4.1]. The proof of this result was just sketched and when the result appeared later in Miao's thesis [42, Theorem 3.11] the full proof was only given for $0 < q < 1$ and the formula only conjectured to hold for $q > 1$.

We show that Miao's conjectured formula is false for $q > 1$ in general by providing an explicit family of counterexamples, see Theorem 2.4.5. We are able to provide new, non-trivial, closed form bounds for the generalised q-dimensions, see Theorem 2.4.7 and also give new conditions which guarantee that the conjectured formula does hold, see Corollary 2.4.12.

2.2 Split binomial sums

In this section we state an important technical result, which is used to provide our family of counterexamples. It states that if for $x > 1$ one looks at the binomial expansion of $(1+x)^k$ and splits the sum in half, then the ratio of the two halves grows exponentially in k.

Lemma 2.2.1. *Let $x > 1$, then*

$$\lim_{j\to\infty}\left(\frac{\sum_{i=j+1}^{2j+1}\binom{2j+1}{i}x^i}{\sum_{i=0}^{j}\binom{2j+1}{i}x^i}\right)^{\frac{1}{2j+1}} = \frac{1+x}{2\sqrt{x}} > 1.$$

Proof. Fix $x > 1$. Since $\binom{2j+1}{i} \leq \binom{2j+1}{j}$ for all $i = 0, \dots, 2j+1$ it follows that $\binom{2j+1}{j} \geq \frac{1}{2j+2}\sum_{i=0}^{2j+1}\binom{2j+1}{i} = \frac{2^{2j+1}}{2j+2}$. Hence

$$\frac{2^{2j+1}x^j}{2j+2} \leq \binom{2j+1}{j}x^j \leq \sum_{i=0}^{j}\binom{2j+1}{i}x^i \leq \sum_{i=0}^{2j+1}\binom{2j+1}{i}x^j = 2^{2j+1}x^j.$$

It follows that

$$\frac{\sum_{i=j+1}^{2j+1}\binom{2j+1}{i}x^i}{\sum_{i=0}^{j}\binom{2j+1}{i}x^i} = \frac{\sum_{i=0}^{2j+1}\binom{2j+1}{i}x^i - \sum_{i=0}^{j}\binom{2j+1}{i}x^i}{\sum_{i=0}^{j}\binom{2j+1}{i}x^i} = \frac{(1+x)^{2j+1}}{\sum_{i=0}^{j}\binom{2j+1}{i}x^i} - 1 \geq \frac{(1+x)^{2j+1}}{2^{2j+1}x^j} - 1$$

and also

$$\frac{\sum_{i=j+1}^{2j+1}\binom{2j+1}{i}x^i}{\sum_{i=0}^{j}\binom{2j+1}{i}x^i} \leq \frac{\sum_{i=0}^{2j+1}\binom{2j+1}{i}x^i}{\sum_{i=0}^{j}\binom{2j+1}{i}x^i} = \frac{(1+x)^{2j+1}}{\sum_{i=0}^{j}\binom{2j+1}{i}x^i} \leq \frac{(2j+2)(1+x)^{2j+1}}{2^{2j+1}x^j}.$$

Since $\frac{1+x}{2\sqrt{x}} > 1$ by the arithmetic-geometric mean inequality the result follows easily. $\square$

2.3 Diagonal systems

We now turn to the first class of IFS we shall study and the corresponding self-affine measure. We begin by introducing the necessary background from [24, 25], starting with diagonal systems.

Definition 2.3.1 (Diagonal System)**.** We say a self-affine IFS is a *diagonal system* if it is an IFS consisting of affine transformations of $\mathbb{R}^2$ whose linear part is a contracting diagonal matrix.

Necessarily the maps that make up diagonal systems are of the form $S_i(x, y) = T_i(x, y)+t_i$, where T_i is a contracting linear map which can be written in matrix form as

$$T_i(x,y) = \begin{pmatrix} \pm c_i & 0 \\ 0 & \pm d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.3.2}$$

where $c_i, d_i \in (0, 1)$ and where $t_i \in \mathbb{R}^2$ is a translation vector.

We also assume that our IFS satisfies the following separation condition.

Definition 2.3.3 (Rectangular open set condition)**.** An IFS on $\mathbb{R}^2$ satisfies the *rectangular open set condition* (ROSC) if $\{S_i((0, 1)^2)\}_{i\in\mathcal{I}}$ are pairwise disjoint subsets of the open unit square $(0, 1)^2$.

We use this separation condition as it was the one originally considered by Fraser in [25]. Although his results and ours seem likely to hold under a weaker separation condition (such as the more general open set condition) we do not pursue this here.

In order to calculate the L^q-spectrum $\tau_\mu(q)$ of self-affine measures supported on diagonal systems Fraser introduced what he termed a *q-modified singular value function*, which can be thought of as an analogue of the traditional singular value function (Definition 1.3.4). To introduce this we begin by defining the projection maps $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$. It may be shown that the projections of the measure μ, namely $\pi_1(\mu) = \mu \circ \pi_1^{-1}$ and $\pi_2(\mu) = \mu \circ \pi_2^{-1}$, are a pair of self-similar measures. It follows from a result of Peres and Solomyak [46] that the L^q-spectra of both of these projected measures, which we denote by $\tau_1(q) := \tau_{\pi_1(\mu)}(q)$ and $\tau_2(q) := \tau_{\pi_2(\mu)}(q)$, exist for $q \geq 0$.

We use the notation introduced in Section 1.7 for sequences over $\mathcal{I}$. For $\mathfrak{i} = (i_1, \ldots, i_k) \in \mathcal{I}^*$ we also write $c(\mathfrak{i}) = c_{i_1} c_{i_2} \cdots c_{i_k}$ and $d(\mathfrak{i}) = d_{i_1} d_{i_2} \cdots d_{i_k}$, where c_i, d_i are as in (2.3.2). In particular, for all $\mathfrak{i} = (i_1, \ldots, i_k) \in \mathcal{I}^*$ it is clear that the singular values $\alpha_1(\mathfrak{i}) = \max\{c(\mathfrak{i}), d(\mathfrak{i})\}$ and $\alpha_2(\mathfrak{i}) = \min\{c(\mathfrak{i}), d(\mathfrak{i})\}$.

Now define $\pi_\mathfrak{i} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\pi_\mathfrak{i} = \begin{cases} \pi_1 & \text{if } c(\mathfrak{i}) \geq d(\mathfrak{i}) \\ \pi_2 & \text{if } c(\mathfrak{i}) < d(\mathfrak{i}) \end{cases}$$

and subsequently define $\tau_\mathfrak{i}(q)$ by $\tau_\mathfrak{i}(q) := \tau_{\pi_\mathfrak{i}(\mu)}(q)$. Note that $\tau_\mathfrak{i}(q)$ is simply the L^q-spectrum of the projection of $\mu|_{S_\mathfrak{i}(F)}$ onto the longest side of the rectangle $S_\mathfrak{i}([0,1]^2)$ and is always equal to either $\tau_1(q)$ or $\tau_2(q)$.

Definition 2.3.4 (q-modified singular value function)**.** For $s \in \mathbb{R}$ and $q \geq 0$, define the *q-modified singular value function*, $\psi^{s,q} : \mathcal{I}^* \to (0, \infty)$ by

$$\psi^{s,q}(\mathtt{i}) = p(\mathtt{i})^q \alpha_1(\mathtt{i})^{\tau_{\mathtt{i}}(q)} \alpha_2(\mathtt{i})^{s-\tau_{\mathtt{i}}(q)}.$$

For each $k \in \mathbb{N}$ we write $\Psi_k^{s,q}$ for the quantity

$$\Psi_k^{s,q} = \sum_{\mathtt{i} \in \mathcal{I}^k} \psi^{s,q}(\mathtt{i}). \tag{2.3.5}$$

It now follows from Lemma 2.2 in [25] and standard properties of submultiplicative sequences that for any $s \in \mathbb{R},\ q \geq 0$ the sequence $\{(\Psi_k^{s,q})^{1/k}\}_{k \in \mathbb{N}}$ is convergent, so we may define a function $P : \mathbb{R} \times [0, \infty) \to [0, \infty)$ by

$$P(s, q) = \lim_{k \to \infty} (\Psi_k^{s,q})^{1/k}.$$

It also follows from Lemma 2.3 in [25] that for any $q \geq 0$, there exists a unique $s' \in \mathbb{R}$ such that $P(s', q) = 1$, so we may define another function, $\gamma : [0, \infty) \to \mathbb{R}$, by $P(\gamma(q), q) = 1$. The importance of this function is the following theorem from [25].

Theorem 2.3.6. [25, Theorem 2.6] *Suppose that μ is generated by a diagonal system and satisfies the ROSC. Then*

$$\tau_\mu(q) = \gamma(q).$$

This tells us that finding a closed form expression for $\tau_\mu(q)$ is equivalent to finding a closed form expression for $\gamma(q)$.

We may approximate $\gamma(q)$ numerically by functions $\gamma_k(q)$, where for each $k \in \mathbb{N}$ we define $\gamma_k(q) : [0, \infty) \to \mathbb{R}$ by

$$\Psi_k^{\gamma_k(q),q} = 1,$$

with $\gamma_k(q) \to \gamma(q)$ as $k \to \infty$. To find a closed form expression Fraser defined functions $\gamma_A, \gamma_B : [0, \infty) \to \mathbb{R}$ by

$$\sum_{i \in \mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} = 1 \tag{2.3.7}$$

and

$$\sum_{i \in \mathcal{I}} p_i^q \ d_i^{\tau_2(q)} \ c_i^{\gamma_B(q)-\tau_2(q)} = 1. \tag{2.3.8}$$

The following lemma tells us about the relationship between γ_A, γ_B and τ_1, τ_2.

Lemma 2.3.9. [25, Lemma 2.9] *Let μ be a self-affine measure generated by a diagonal system and $q \geq 0$. Then either*

$$\max\{\gamma_A(q), \gamma_B(q)\} \leq \tau_1(q) + \tau_2(q)$$

or

$$\min\{\gamma_A(q), \gamma_B(q)\} \geq \tau_1(q) + \tau_2(q).$$

This lemma is particularly helpful as it allows us to state Fraser's main result on closed form expressions from [25].

Theorem 2.3.10. [25, Theorem 2.10] *Let μ be be a self-affine measure generated by a diagonal system and $q \geq 0$.*

If $\max\{\gamma_A(q), \gamma_B(q)\} \leq \tau_1(q) + \tau_2(q)$ *then*

$$\gamma(q) = \max\{\gamma_A(q), \gamma_B(q)\}.$$

If $\min\{\gamma_A(q), \gamma_B(q)\} \geq \tau_1(q) + \tau_2(q),$ *then*

$$\tau_1(q) + \tau_2(q) \leq \gamma(q) \leq \min\{\gamma_A(q), \gamma_B(q)\}$$

and if either

$$\sum_{i \in \mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q) - \tau_1(q)} \ \log(c_i/d_i) \geq 0 \tag{2.3.11}$$

or

$$\sum_{i \in \mathcal{I}} p_i^q \ d_i^{\tau_2(q)} \ c_i^{\gamma_B(q) - \tau_2(q)} \ \log(d_i/c_i) \geq 0 \tag{2.3.12}$$

then $\gamma(q) = \min\{\gamma_A(q), \gamma_B(q)\}$.

The fact that we only have an inequality involving $\gamma(q)$ when $\min\{\gamma_A(q), \gamma_B(q)\} \geq \tau_1(q) + \tau_2(q)$, combined with the observation that the conditions (2.3.11) and (2.3.12) do not look especially natural, led Fraser to ask the following question.

Question 2.3.13. [25, Question 2.14]

If $\min\{\gamma_A(q), \gamma_B(q)\} \geq \tau_1(q) + \tau_2(q)$ and neither

$$\sum_{i \in \mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q) - \tau_1(q)} \ \log(c_i/d_i) \geq 0$$

nor

$$\sum_{i \in \mathcal{I}} p_i^q \ d_i^{\tau_2(q)} \ c_i^{\gamma_B(q) - \tau_2(q)} \ \log(d_i/c_i) \geq 0$$

are satisfied, is it still true that

$$\gamma(q) = \min\{\gamma_A(q), \gamma_B(q)\}?$$

By presenting a family of counterexamples we shall answer this question in the negative.

In particular we provide a family of diagonal systems consisting of two maps equipped with the Bernoulli-(1/2, 1/2) measure such that

$$\gamma(q) < \min\{\gamma_A(q), \gamma_B(q)\}$$

for all $q > 1$.

2.3.1 A family of counterexamples

We now give examples answering Question 2.3.13 in the negative. We require a family of measures such that the two conditions in Theorem 2.3.10 fail. At the same time we also need to ensure that they are simple enough to allow us to estimate $\Psi_k^{s,q}$ in (2.3.5) effectively. We prove the following result, which states that, for a certain explicit family of self-affine measures generated by diagonal systems, $\tau_\mu(q)$ is not equal to either $\gamma_A(q)$ or $\gamma_B(q)$ for all $q > 1$. Lemma 2.2.1 will be of key importance in establishing this result.

Theorem 2.3.14. *Let c, d be such that $c > d > 0$ and $c + d \leq 1$. Let μ be the self-affine measure defined by the probability vector $(1/2, 1/2)$ and the diagonal system consisting of the two maps, S_1 and S_2, where*

$$S_1(x,y) = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textit{and} \qquad S_2(x,y) = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1-d \\ 1-c \end{pmatrix}.$$

and where $\{S_1, S_2\}$ satisfies the ROSC. Then, for $q > 1$,

$$\gamma(q) < \min\{\gamma_A(q), \gamma_B(q)\}.$$

More precisely, for $q > 1$, $\gamma_A(q) = \gamma_B(q) < 0$ and, writing s to denote this common *value,*

$$\gamma(q) \leq s - \frac{2\log\left(\frac{2(d/c)^{s/2}}{(d/c)^s+1}\right)}{\log(cd)}. \tag{2.3.15}$$

Proof. Let $q > 1$. The projection of μ to the horizontal and vertical axes are self-similar measures satisfying the open set condition, so it follows by symmetry and (1.6.9) that $\tau_1(q) = \tau_2(q)$ and their common value is given by the unique s satisfying

$$2^{-q}c^s + 2^{-q}d^s = 1. \tag{2.3.16}$$

Furthermore if we consider (2.3.7) and (2.3.8) then as $\tau_1(q) = \tau_2(q)$ it is easy to see that in our setting these expressions are the same, giving $\gamma_A(q) = \gamma_B(q)$. Finally if we then compare (2.3.7) and (2.3.8) with (2.3.16) it is easy to see that in fact $\tau_1(q) = \tau_2(q) = \gamma_A(q) = \gamma_B(q) = s < 0$.

Let k be odd. We may write $\Psi_k^{s,q}$ as

$$\begin{aligned} \Psi_k^{s,q} &= \sum_{\mathtt{i}\in\mathcal{I}^k} p_{\mathtt{i}}^q \ \alpha_1(\mathtt{i})^{\tau_{\mathtt{i}}(q)} \ \alpha_2(\mathtt{i})^{s-\tau_{\mathtt{i}}(q)} \\ &= \sum_{\mathtt{i}\in\mathcal{I}^k} 2^{-kq} \ \alpha_1(\mathtt{i})^s, \end{aligned} \tag{2.3.17}$$

using the fact that $p = 1/2$ and $s = \tau_1(q) = \tau_2(q)$. Since the maps S_1 and S_2 commute, we can write each $S_{\mathtt{i}}$ ($\mathtt{i} \in \mathcal{I}^k$) as $S_{\mathtt{i}} = S_1^i \circ S_2^{k-i}$ where $i \in [0,k]$ is the number of times S_1 was used in the composition of $S_{\mathtt{i}}$. For such maps, since $c > d$,

$$\alpha_1(\mathtt{i}) = c^{\max\{i,k-i\}} \times d^{\min\{i,k-i\}}$$

and we can re-express (2.3.17) as

$$\Psi_k^{s,q} = X_k^q + Y_k^q,$$

where

$$X_k^q = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} 2^{-kq} \left(c^{k-i} d^i\right)^s$$

and

$$Y_k^q = \sum_{i=\lceil k/2 \rceil}^{k} \binom{k}{i} 2^{-kq} \left(d^{k-i} c^i\right)^s.$$

We now turn to the ratio $X_k^q/(1 - X_k^q)$. By the binomial theorem and the definition of $s = \gamma_A(q)$,

$$\sum_{i=0}^{k} \binom{k}{i} 2^{-kq} \left(c^{k-i} d^i\right)^s = \left(2^{-q} c^{\gamma_A(q)} + 2^{-q} d^{\gamma_A(q)}\right)^k = 1^k = 1 \tag{2.3.18}$$

and therefore

$$\frac{X_k^q}{1 - X_k^q} = \frac{\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} 2^{-kq} \left(c^{k-i} d^i\right)^s}{\sum_{i=\lceil k/2 \rceil}^{k} \binom{k}{i} 2^{-kq} \left(c^{k-i} d^i\right)^s}.$$

We may rearrange (and cancel a factor $2^{-kq} c^{ks}$) to give

$$\frac{X_k^q}{1 - X_k^q} = \frac{\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} \left((d/c)^s\right)^i}{\sum_{i=\lceil k/2 \rceil}^{k} \binom{k}{i} ((d/c)^s)^i}. \tag{2.3.19}$$

As $c > d$ and as $s < 0$ we have $(d/c)^s > 1$. Thus by Lemma 2.2.1,

$$\left(\frac{X_k^q}{1 - X_k^q}\right)^{1/k} \to \frac{2(d/c)^{s/2}}{(d/c)^s + 1} =: \delta \in (0, 1) \tag{2.3.20}$$

as $k \to \infty$. Now consider $(1 - X_k^q)^{1/k}$. It is easy to see from the definition of X_k^q and (2.3.18) that $0 \le X_k^q \le 1$, which gives $0 \le 1 - X_k^q \le 1$. This implies $(1 - X_k^q)^{1/k} \to 1$ unless $\liminf_{k\to\infty}(1 - X_k^q) = 0$. But $\liminf_{k\to\infty}(1 - X_k^q) = 0$ would imply

$$\left(\frac{X_k^q}{1 - X_k^q}\right)^{1/k} \ge 1.$$

for some sufficiently large values of k, contradicting (2.3.20). Therefore $(1 - X_k^q)^{1/k} \to 1$, which together with (2.3.20) gives $(X_k^q)^{1/k} \to \delta$ as $k \to \infty$. By similar reasoning we deduce the same result for Y_k^q, in particular

$$\begin{aligned}
\frac{Y_k^q}{1-Y_k^q} &= \frac{\sum_{i=\lceil k/2\rceil}^{k} \binom{k}{i} 2^{-kq} \left(d^{k-i}c^i\right)^s}{\sum_{i=0}^{\lfloor k/2\rfloor} \binom{k}{i} 2^{-kq} \left(d^{k-i}c^i\right)^s} = \frac{\sum_{i=\lceil k/2\rceil}^{k} \binom{k}{i} \left(d^{k-i}c^i\right)^s}{\sum_{i=0}^{\lfloor k/2\rfloor} \binom{k}{i} \left(d^{k-i}c^i\right)^s} \\
&= \frac{\sum_{l=0}^{\lfloor k/2\rfloor} \binom{k}{l} \left((d/c)^s\right)^l}{\sum_{l=\lceil k/2\rceil}^{k} \binom{k}{l} \left((d/c)^s\right)^l} \qquad (2.3.21)
\end{aligned}$$

(this follows from relabelling the summation by $l = k - i$ and using the fact that $\binom{k}{k-l} = \binom{k}{l}$).

Note (2.3.21) gives the same expression for $Y_k^q/(1-Y_k^q)$ that we found for $X_k^q/(1-X_k^q)$ in (2.3.19), so by the same reasoning $(Y_k^q)^{1/k} \to \delta$ as $k \to \infty$ as well. Therefore

$$P(s,q) = \lim_{k\to\infty} \left(\Psi_k^{s,q}\right)^{1/k} = \lim_{k\to\infty} \left(X_k^q + Y_k^q\right)^{1/k} = \delta < 1$$

and by definition of $P(t,q)$ and $\gamma(q)$

$$P(\gamma(q), q) = 1 > \delta = P(s,q).$$

Since $P(t,q)$ is decreasing in t this implies $\gamma(q) < s = \gamma_A(q) = \gamma_B(q)$, which is enough to show that $\gamma(q) < \min\{\gamma_A(q), \gamma_B(q)\}$. We can upgrade this result to get the stated quantitative upper bound (2.3.15) by examining the function $P(t,q)$ more closely. For $k \geq 1$ and $\mathfrak{i} \in \mathcal{I}^k$, $\alpha_1(\mathfrak{i}) \geq (cd)^{k/2}$ and therefore, for $\varepsilon = s - \gamma(q) > 0$,

$$\begin{aligned}
\delta = P(s,q) = \lim_{k\to\infty} \left(\sum_{\mathfrak{i}\in\mathcal{I}^k} 2^{-kq}\ \alpha_1(\mathfrak{i})^{\gamma(q)+\varepsilon}\right)^{1/k} &\geq \lim_{k\to\infty} \left((cd)^{\varepsilon k/2} \sum_{\mathfrak{i}\in\mathcal{I}^k} 2^{-kq}\ \alpha_1(\mathfrak{i})^{\gamma(q)}\right)^{1/k} \\
&= (cd)^{\varepsilon/2} P(\gamma(q), q) \\
&= (cd)^{\varepsilon/2}.
\end{aligned}$$

Hence

$$s - \gamma(q) = \varepsilon \geq \frac{2\log\delta}{\log(cd)},$$

which proves the theorem. $\square$

2.3.2 New examples of phase transitions

Here we record a simple consequence of Theorem 2.3.14 relating to phase transitions.

The differentiability of the L^q-spectrum is important and has many interesting consequences. Key among these is the result of Ngai [43] that we saw in Section 1.5, namely if $\tau'_\mu(1)$ exists then $-\tau'_\mu(1)$ gives the Hausdorff dimension of μ. We can use Theorem 2.3.14 to provide examples of behaviour relating to higher order phase transitions at $q = 1$. We are unaware of any other method for constructing such examples.

Theorem 2.3.22. *Let μ be a planar self-affine measure μ as in Theorem 2.3.14. Then τ_μ is differentiable at $q = 1$ but not analytic in any neighbourhood of $q = 1$.*

Proof. As the functions τ_1, τ_2 are the L^q-spectra of the measures $\pi_1\mu, \pi_2\mu$ and these measures are self-similar and satisfy the open set condition, it follows that they are real analytic on $(0, \infty)$, see [16, Chapter 17] (in particular they are differentiable at $q = 1$). We can therefore apply Theorem 2.12 from [25] and conclude that the function $\gamma(q)$ is differentiable at $q = 1$, so that $\tau_\mu = \gamma$ is differentiable at $q = 1$.

Observe that the function $\gamma_A = \gamma_B$ (where γ_A and γ_B are as in (2.3.7) and (2.3.8)) is also real analytic on $(0, \infty)$, since it inherits analyticity from τ_1, τ_2 via the analytic implicit function theorem (here we are using the uniqueness part of the analytic function theorem as this guarantees that $\gamma_A = \gamma_B$ is the only solution of (2.3.7) and (2.3.8)). We know that $\gamma(q) = \gamma_A(q) = \gamma_B(q)$ for $q \in [0, 1]$ but $\gamma(q) < \gamma_A(q) = \gamma_B(q)$ for $q > 1$, see Theorem 2.3.14. It follows that $\tau_\mu = \gamma$ cannot be analytic on any neighbourhood of $q = 1$. $\square$

Given the intriguing phenomena observed in Theorem 2.3.22, a natural next step would be to investigate the differentiability of τ_μ at $q = 1$ further.

Question 2.3.23. How many derivatives does $\tau_\mu = \gamma$ have at $q = 1$ for the measures μ considered in Theorem 2.3.14?

2.3.3 New closed form lower bounds

We now know that $\gamma(q)$ is not in general given by either the maximum or minimum of $\gamma_A(q)$ and $\gamma_B(q)$. However, by developing a quantitative version of the argument in [25] used to prove Theorem 2.3.10 we are able to provide new closed form lower bounds for $\gamma(q)$ for all planar diagonal systems. Given $x \in \mathbb{R}$ we write $x^+ = \max\{x, 0\}$.

Theorem 2.3.24. *Let μ be a planar self-affine measure generated by a diagonal system and let $q \geq 0$. Then*

$$\gamma(q) \geq \max\{L_A(q), L_B(q)\} \tag{2.3.25}$$

where

$$L_A(q) = \gamma_A(q) - \left(\Big(\gamma_A(q) - \tau_1(q) - \tau_2(q)\Big) \frac{\sum_{i \in \mathcal{I}} p_i^q \; c_i^{\tau_1(q)} \; d_i^{\gamma_A(q) - \tau_1(q)} \log(c_i/d_i)}{\sum_{i \in \mathcal{I}} p_i^q \; c_i^{\tau_1(q)} \; d_i^{\gamma_A(q) - \tau_1(q)} \log(c_i)} \right)^+$$

and

$$L_B(q) = \gamma_B(q) - \left(\Big(\gamma_B(q) - \tau_1(q) - \tau_2(q)\Big) \frac{\sum_{i \in \mathcal{I}} p_i^q \; d_i^{\tau_2(q)} \; c_i^{\gamma_B(q) - \tau_2(q)} \log(d_i/c_i)}{\sum_{i \in \mathcal{I}} p_i^q \; d_i^{\tau_2(q)} \; c_i^{\gamma_B(q) - \tau_2(q)} \log(d_i)} \right)^+ .$$

In particular,

$$\frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i)} < 1$$

and

$$\frac{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{\tau_2(q)} \ c_i^{\gamma_B(q)-\tau_2(q)} \log(d_i/c_i)}{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{\tau_2(q)} \ c_i^{\gamma_B(q)-\tau_2(q)} \log(d_i)} < 1,$$

which ensures that (2.3.25) *provides a strictly better bound than* $\gamma(q) \geq \tau_1(q) + \tau_2(q)$ *in the case when* $\gamma(q) \leq \min\{\gamma_A(q), \gamma_B(q)\}$*.*

Proof. We prove that $\gamma(q) \geq L_A(q)$. The inequality $\gamma(q) \geq L_B(q)$ follows by an analogous argument which we omit. Let $\{\theta_i\}_{i\in\mathcal{I}}$ denote an arbitrary probability vector and for each $k \in \mathbb{N}$ define a number $n(k) \in \mathbb{N}$ by

$$n(k) = \sum_{i\in\mathcal{I}} \lfloor \theta_i k \rfloor, \tag{2.3.26}$$

which satisfies $k - |\mathcal{I}| \leq n(k) \leq k$. We consider the $n(k)$th iteration of $\mathcal{I}$ and define

$$\mathcal{J}_k = \left\{ \mathtt{j} = (j_1, \ldots, j_{n(k)}) \in \mathcal{I}^{n(k)} : \#\{m : j_m = i\} = \lfloor \theta_i k \rfloor \text{ for each } i \in \mathcal{I} \right\}, \tag{2.3.27}$$

where the size of $\mathcal{J}_k$ is given by

$$|\mathcal{J}_k| = \frac{n(k)!}{\prod_{i\in\mathcal{I}} \lfloor \theta_i k \rfloor !}.$$

We also define numbers c, d and p (for which we suppress the dependence on k) by

$$c = \prod_{i\in\mathcal{I}} c_i^{\lfloor \theta_i k \rfloor}, \qquad d = \prod_{i\in\mathcal{I}} d_i^{\lfloor \theta_i k \rfloor}, \qquad p = \prod_{i\in\mathcal{I}} p_i^{\lfloor \theta_i k \rfloor}.$$

First assume that $\prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i}$, which implies that $c > d$ for k sufficiently large.

Indeed

$$c = \prod_{i \in \mathcal{I}} c_i^{\lfloor \theta_i k \rfloor} \geq \left(\prod_{i \in \mathcal{I}} c_i^{\theta_i} \right)^k$$

and

$$d = \prod_{i \in \mathcal{I}} d_i^{\lfloor \theta_i k \rfloor} \leq \left(\prod_{i \in \mathcal{I}} d_i^{\theta_i} \right)^k \left(\prod_{i \in \mathcal{I}} d_i \right)^{-1}$$

and therefore $c > d$ for all

$$k > \frac{-\log \left(\prod_{i \in \mathcal{I}} d_i \right)}{\log \left(\left(\prod_{i \in \mathcal{I}} c_i^{\theta_i} \right) / \left(\prod_{i \in \mathcal{I}} d_i^{\theta_i} \right) \right)}.$$

Therefore, for all sufficiently large k, $\mathfrak{i} \in \mathcal{J}_k$ and $s \in \mathbb{R}$,

$$\psi^{s,q}(\mathfrak{i}) = p^q \ c^{\tau_1(q)} \ d^{s-\tau_1(q)}. \tag{2.3.28}$$

By definition of c, d and p we may write this as

$$\psi^{s,q}(\mathfrak{i}) = \prod_{i \in \mathcal{I}} \left(p_i^q \ c_i^{\tau_1(q)} \ d_i^{s-\tau_1(q)} \right)^{\lfloor \theta_i k \rfloor}.$$

We now introduce a form of Stirling's approximation which states that

$$n! \backsim n^n e^{-n} \sqrt{2\pi n}.$$

It follows that for $n \in \mathbb{N}$ sufficiently large

$$n \log n - n \leq \log n! \leq n \log n - n + \log n. \tag{2.3.29}$$

Using (2.3.29) as well as (2.3.27) and (2.3.28) we find that for k sufficiently large

$$\begin{aligned}
\log\left(\Psi^{s,q}_{n(k)}\right) &\geq \log\left(\sum_{\mathfrak{i}\in\mathcal{J}_k} \psi^{s,q}(\mathfrak{i})\right) \\
&= \log\left(|\mathcal{J}_k| \prod_{i\in\mathcal{I}} \left(p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}\right)^{\lfloor\theta_i k\rfloor}\right) \\
&= \left(\log n(k)! - \sum_{i\in\mathcal{I}} \log\lfloor\theta_i k\rfloor! + \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \log\left(p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}\right)\right) \\
&\geq \Bigg(n(k)\log n(k) - n(k) - \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \log\lfloor\theta_i k\rfloor + \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \\
&\qquad - \sum_{i\in\mathcal{I}} \log\lfloor\theta_i k\rfloor + \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \log\left(p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}\right)\Bigg),
\end{aligned}$$

where the last line follows from (2.3.29). Introducing an exponent of $1/n(k)$ and using (2.3.26) gives

$$\begin{aligned}
\log\left(\Psi^{s,q}_{n(k)}\right)^{1/n(k)} &\geq \frac{1}{n(k)}\Bigg(n(k)\log n(k) - \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \log k - \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \log\theta_i \\
&\qquad - \sum_{i\in\mathcal{I}} \log\lfloor\theta_i k\rfloor + \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \log\left(p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}\right)\Bigg) \\
&\geq \frac{1}{n(k)}\Bigg(n(k)\log n(k) - n(k)\log k - \sum_{i\in\mathcal{I}} \log\lfloor\theta_i k\rfloor \\
&\qquad + \sum_{i\in\mathcal{I}} \lfloor\theta_i k\rfloor \log\left(\frac{p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}}{\theta_i}\right)\Bigg) \\
&\geq \log\left(\frac{k-|\mathcal{I}|}{k}\right) - \frac{1}{k-|\mathcal{I}|}\sum_{i\in\mathcal{I}} \log\theta_i k \\
&\qquad + \sum_{i\in\mathcal{I}} \frac{\lfloor\theta_i k\rfloor}{n(k)} \log\left(\frac{p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}}{\theta_i}\right),
\end{aligned}$$

where the last line uses $k-|\mathcal{I}| \leq n(k)$. As

$$\frac{\theta_i k - 1}{k} \leq \frac{\lfloor\theta_i k\rfloor}{n(k)} \leq \frac{\theta_i k}{k-|\mathcal{I}|}$$

for each $i \in \mathcal{I}$ it follows that

$$\lim_{k\to\infty} \log \left(\Psi^{s,q}_{n(k)}\right)^{1/n(k)} \geq \sum_{i\in\mathcal{I}} \theta_i \log \left(\frac{p_i^q \; c_i^{\tau_1(q)} \; d_i^{s-\tau_1(q)}}{\theta_i} \right).$$

If

$$\sum_{i\in\mathcal{I}} \theta_i \log \left(\frac{p_i^q \; c_i^{\tau_1(q)} \; d_i^{s-\tau_1(q)}}{\theta_i} \right) \geq 0, \tag{2.3.30}$$

then

$$P(s,q) = \lim_{k\to\infty} \left(\Psi^{s,q}_{n(k)}\right)^{1/n(k)} \geq 1$$

and therefore $\gamma(q) \geq s$.

Second, assume that $\prod_{i\in\mathcal{I}} c_i^{\theta_i} < \prod_{i\in\mathcal{I}} d_i^{\theta_i}$. In this case, a completely analogous argument proves that if

$$\sum_{i\in\mathcal{I}} \theta_i \log \left(\frac{p_i^q \; d_i^{\tau_2(q)} \; c_i^{s-\tau_2(q)}}{\theta_i} \right) \geq 0, \tag{2.3.31}$$

then $P(s,q) \geq 1$ and so $\gamma(q) \geq s$.

Finally, if $\prod_{i\in\mathcal{I}} c_i^{\theta_i} = \prod_{i\in\mathcal{I}} d_i^{\theta_i}$ then we cannot guarantee that $c > d$ or $d > c$ for all k sufficiently large. We can however conclude that we must have either $c \geq d$ or $d \geq c$ (or both) for infinitely many k, so by choosing an appropriate subsequence we can reduce to one of the above two cases. Since we do not know which case we are in ($c \geq d$ or $d \geq c$) we require that both (2.3.30) and (2.3.31) hold. Putting the above three cases together we have shown that

$$
\begin{aligned}
\gamma(q) \geq \sup \Bigg\{ s : & \text{ there exists a probability vector } \{\theta_i\}_{i\in\mathcal{I}} \text{ such that either} \\
& (1) \prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i} \text{ and } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}}{\theta_i}\right) \geq 0 \\
& \text{or } (2) \prod_{i\in\mathcal{I}} c_i^{\theta_i} < \prod_{i\in\mathcal{I}} d_i^{\theta_i} \text{ and } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ d_i^{\tau_2(q)}\ c_i^{s-\tau_2(q)}}{\theta_i}\right) \geq 0 \\
& \text{or } (3) \prod_{i\in\mathcal{I}} c_i^{\theta_i} = \prod_{i\in\mathcal{I}} d_i^{\theta_i} \text{ and both } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}}{\theta_i}\right) \geq 0 \\
& \text{and } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ d_i^{\tau_2(q)}\ c_i^{s-\tau_2(q)}}{\theta_i}\right) \geq 0 \Bigg\}.
\end{aligned}
$$

In the above we have the freedom to choose any probability vector $\{\theta_i\}_{i\in\mathcal{I}}$. Perhaps the most natural choice (suggested by Lagrange multipliers) is to take

$$\{\theta_i\}_{i\in\mathcal{I}} = \left\{p_i^q\ c_i^{\tau_1(q)}\ d_i^{\gamma_A(q)-\tau_1(q)}\right\}_{i\in\mathcal{I}}$$

(this is a probability vector by definition of γ_A). We now let $s = \gamma_A(q) - \varepsilon$ for $\varepsilon \geq 0$. We want to see how small we can make ε (ideally we want $\varepsilon = 0$) such that the two conditions (2.3.30) and (2.3.31) hold simultaneously. We need both of these conditions to hold simultaneously as we do not know if $\prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i}$ or $\prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i}$. The first holds trivially, since

$$\sum_{i\in\mathcal{I}} p_i^q\ c_i^{\tau_1(q)}\ d_i^{\gamma_A(q)-\tau_1(q)} \log\left(\frac{p_i^q\ c_i^{\tau_1(q)}\ d_i^{\gamma_A(q)-\varepsilon-\tau_1(q)}}{p_i^q\ c_i^{\tau_1(q)}\ d_i^{\gamma_A(q)-\tau_1(q)}}\right) = \sum_{i\in\mathcal{I}} p_i^q\ c_i^{\tau_1(q)}\ d_i^{\gamma_A(q)-\tau_1(q)} \log(d_i^{-\varepsilon}) \geq 0.$$

For the second to hold, we require

$$\sum_{i\in\mathcal{I}} p_i^q\ c_i^{\tau_1(q)}\ d_i^{\gamma_A(q)-\tau_1(q)} \log\left(\frac{p_i^q\ d_i^{\tau_2(q)}\ c_i^{\gamma_A(q)-\varepsilon-\tau_2(q)}}{p_i^q\ c_i^{\tau_1(q)}\ d_i^{\gamma_A(q)-\tau_1(q)}}\right) \geq 0.$$

Rearranging this, we see that this is equivalent to having

$$\varepsilon \geq \Big(\gamma_A(q) - \tau_1(q) - \tau_2(q)\Big) \frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i)}. \tag{2.3.32}$$

We see that when Fraser's original condition from Theorem 2.10 in [25] holds, namely if

$$\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i/d_i) \geq 0,$$

then right hand side of (2.3.32) is negative so we may take $\varepsilon = 0$. Otherwise we use the bound for ε given in (2.3.32). Putting these two cases together gives

$$\gamma(q) \geq \gamma_A(q) - \left(\Big(\gamma_A(q) - \tau_1(q) - \tau_2(q)\Big) \frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i)}\right)^+ .$$

Finally

$$\frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i)} = 1 - \frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{\tau_1(q)} \ d_i^{\gamma_A(q)-\tau_1(q)} \log(c_i)} < 1,$$

so our lower bound is indeed an improvement on

$$\gamma(q) \geq \tau_1(q) + \tau_2(q)$$

in the case when $\gamma(q) \leq \min\{\gamma_A(q), \gamma_B(q)\}$. $\square$

2.3.4 An example

Here we present an example of a diagonal system satisfying the assumptions of Theorem 2.3.14 where we take $c = 3/4$ and $d = 1/4$. We know from Theorem 2.3.14 that $\tau_\mu(q) = \gamma(q)$ is not given by the maximum or minimum of $\gamma_A(q)$ and $\gamma_B(q)$ for $q > 1$. It is therefore natural to seek bounds for the L^q-spectrum.

Let $q > 1$. Focusing on upper bounds, Theorem 2.3.10 implies that, for $q > 1$, $\gamma_A(q) = \gamma_B(q) = \tau_1(q) = \tau_2(q) = s < 0$, where s is the solution of

$$2^{-q}c^s + 2^{-q}d^s = 1,$$

and

$$\gamma(q) \leq s - \frac{2\log\left(\frac{2(d/c)^{s/2}}{(d/c)^s+1}\right)}{\log(cd)}.$$

Concerning lower bounds, Theorem 2.3.24 implies that

$$\gamma(q) \geq \max\{L_A(q), L_B(q)\} = s\left(2 - \frac{c^s\log(d) + d^s\log(c)}{c^s\log(c) + d^s\log(d)}\right).$$

There are also a couple of trivial lower bounds. Since $\gamma(0) = 1$ (the box dimension of the support of μ), $\gamma(1) = 0$, and γ is necessarily convex, it follows that $1 - q$ is a lower bound for $\tau_\mu(q)$. We also know that $\tau_1(q)+\tau_2(q)$ is a lower bound for $\tau_\mu(q)$, see a remark following [25, Question 2.14]. Figure 2.1 shows a plot of these bounds for $q \in [1, 20]$. We see that our new lower bound, $\max\{L_A(q), L_B(q)\}$ is a strict improvement on the lower bound of $1 - q$ outside of the the range $(1.7, 9.3)$.

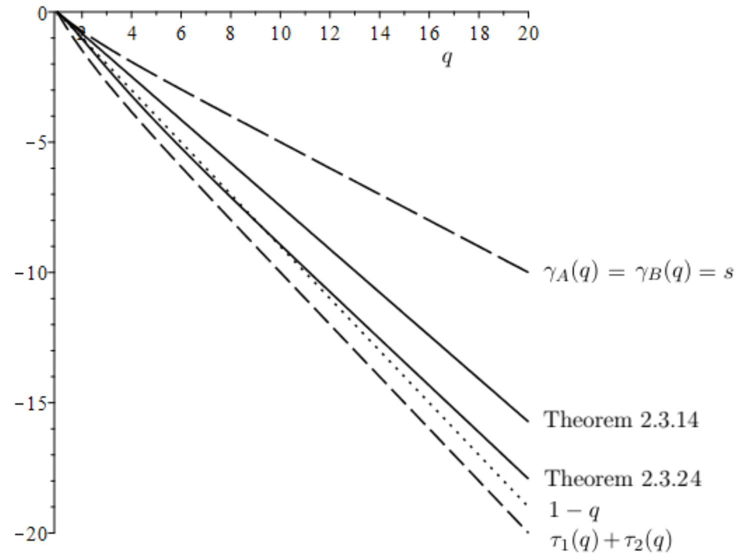

Figure 2.1: Graph of our new upper and lower bounds for the L^q-spectrum (solid lines), labelled by the theorem they come from. For reference we also show graphs of the previously known upper bound $\min\{\gamma_A(q), \gamma_B(q)\}$ (long dash) and the previously known lower bound $\tau_1(q) + \tau_2(q)$ (short dash), as well as the lower bound $1 - q$, which is specific to this setting (dots).

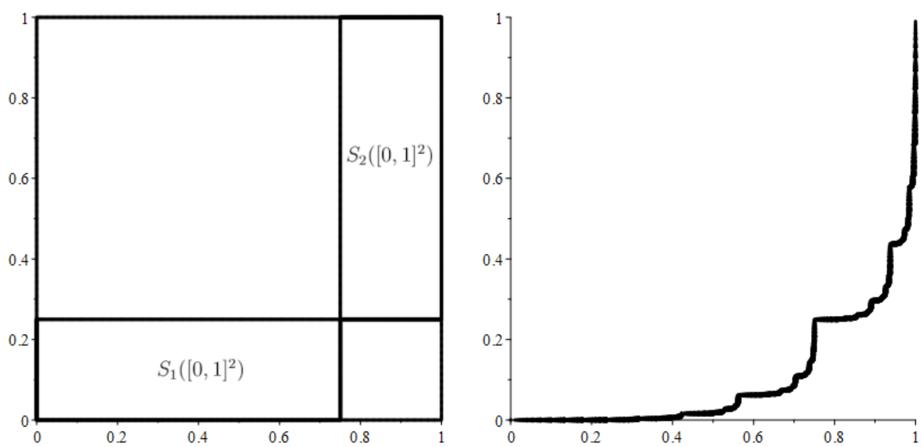

Figure 2.2: Left: images of the unit square under the two maps used above. Right: the associated self-affine set.

2.4 Generalised q-dimensions in the generic setting

In [18] Falconer and Miao studied self-affine sets and measures generated by IFSs consisting of upper-triangular matrices. This paper was mainly concerned with dimensions of self-affine *sets*, but towards the end of the paper they stated a closed form expression for the generalised q-dimensions (Definition 1.6.8) in the measure setting. We show that in fact their formula does *not* always hold when $q > 1$.

In order to calculate the generalised q-dimensions of self-affine measures μ associated with contracting upper triangular matrices $T_1, \dots, T_N$ and probabilities $p_1, \dots, p_N$ Falconer and Miao [18] studied the quantity $d_q(T_1, \dots, T_N, \mu)$.

Definition 2.4.1. For each $q \geq 0$ ($q \neq 1$) we define $d_q(T_1, \dots, T_N, \mu)$ to be the unique t satisfying

$$\lim_{k \to \infty} \left(\sum_{\mathtt{i} \in \mathcal{I}^k} \phi^t(T_{\mathtt{i}})^{1-q} p_{\mathtt{i}}^q \right)^{1/k} = 1,$$

where ϕ^t denotes the singular value function (see Definition 1.3.4).

This approach was introduced in [14] where it was shown that for $q \in (1, 2)$ the generalised q-dimensions of a self-affine measure is generically given by $d_q(T_1, \dots, T_N, \mu)$ in an appropriate sense. See [15] where further results along these lines were obtained for almost self-affine measures. It is therefore of great interest to provide closed form expressions for $d_q(T_1, \dots, T_N, \mu)$ or at least to be able to estimate it effectively. We state the result using our notation and only in the planar case, although it is possible to apply our methods to the higher dimensional setting.

Let $T_1, \dots, T_N$ be a collection of contracting non-singular 2×2 upper triangular matrices and let c_i, d_i denote the diagonal entries of the ith matrix. Define a function P_0 :

$[0,2] \times [0,1) \cup (1,\infty) \to [0,\infty)$ by

$$P_0(t,q) = \begin{cases} \max\left\{\sum_{i=1}^N p_i^q \ c_i^{t(1-q)}, \sum_{i=1}^N p_i^q \ d_i^{t(1-q)}\right\}, & 0 \le t < 1 \\ \max\left\{\sum_{i=1}^N p_i^q \left(c_i^{2-t}(c_i \ d_i)^{t-1}\right)^{1-q}, \sum_{i=1}^N p_i^q \left(d_i^{2-t}(c_i \ d_i)^{t-1}\right)^{1-q}\right\}, & 1 \le t \le 2 \end{cases} \tag{2.4.2}$$

and, for each $q \in [0,1) \cup (1,\infty)$, let $u_0(q)$ be defined by $P_0(u_0(q),q) = 1$, provided a solution exists, otherwise simply let $u_0(q) = 2$.

Theorem 2.4.3. [18, Theorem 4.1] *Let μ be a planar self-affine measure generated by an IFS of upper triangular matrices as above. Then for $q \in [0,1)$*

$$d_q(T_1, \ldots, T_N, \mu) = u_0(q).$$

In the paper [18], this result was suggested to hold for all $q \ge 0$ $(q \ne 1)$. The result appeared again in Miao's PhD thesis [42, Theorem 3.11] in which he observed that, in fact, he could only establish the result for $q \in [0,1)$. Miao conjectured that the result should still hold for $q > 1$, see discussion leading up to [42, Theorem 3.11]. Our main result in this section, which is essentially an analogue of Theorem 2.3.14 adapted to this situation, proves that Theorem 2.4.3, does *not* hold for $q > 1$ in general.

The approach in [18, 42] does however provide a lower bound for $d_q(T_1, \ldots, T_N, \mu)$ for $q > 1$, that is, for all $q > 1$,

$$d_q(T_1, \ldots, T_N, \mu) \ge u_0(q).$$

2.4.1 A family of counterexamples relating to generalised q-dimensions

Before looking at the range $q > 1$ we observe that a better lower bound than $u_0(q)$ is available simply by changing the maximum to a minimum in (2.4.2), which is natural for $q > 1$. We define $P_0^* : [0,2] \times [0,1) \cup (1,\infty) \to [0,\infty)$ by $P_0^*(t,q) = P_0(t,q)$ for $q \in [0,1)$ and for $q > 1$ by

$$P_0^*(t,q) = \begin{cases} \min\left\{\sum_{i=1}^N p_i^q \ c_i^{t(1-q)}, \sum_{i=1}^N p_i^q \ d_i^{t(1-q)}\right\}, & 0 \le t < 1 \\ \min\left\{\sum_{i=1}^N p_i^q \left(c_i^{2-t}(c_i \ d_i)^{t-1}\right)^{1-q}, \sum_{i=1}^N p_i^q \left(d_i^{2-t}(c_i \ d_i)^{t-1}\right)^{1-q}\right\}, & 1 \le t \le 2. \end{cases}$$

Let $u(q)$ be defined by $P_0^*(u(q),q) = 1$, provided a solution exists and otherwise simply let $u(q) = 2$. Note that $u(q) = u_0(q)$ for $q \in [0,1)$ and $u(q) \ge u_0(q)$ for $q > 1$ with strict inequality a possibility. This inequality comes from the fact that the functions that we are taking the maximum or minimum of are *increasing* in t for $q > 1$. We expect that when searching for a closed form expression for $d_q(T_1,\dots,T_N,\mu)$ for $q > 1$, Miao [42] was thinking of $u(q)$ rather than $u_0(q)$.

Lemma 2.4.4. *For all $q \ge 0$ ($q \neq 1$) we have*

$$d_q(T_1,\dots,T_N,\mu) \ge u(q).$$

Proof. It suffices to assume $q > 1$ since for $q < 1$ this result is covered by [18, 42]. Write $\alpha_1(\mathtt{i}) \ge \alpha_2(\mathtt{i})$ for the singular values of the matrix $T_\mathtt{i}$ and let ϕ denote the singular value function (Definition 1.3.4). Firstly suppose $0 \le u(q) < 1$, so

$$\sum_{\mathtt{i}\in\mathcal{I}^k} \phi^{u(q)}(T_\mathtt{i})^{1-q} p_\mathtt{i}^q = \sum_{\mathtt{i}\in\mathcal{I}^k} \alpha_1(\mathtt{i})^{u(q)(1-q)} p_\mathtt{i}^q.$$

By definition $\alpha_1(\mathtt{i}) = \max\{c_\mathtt{i}, d_\mathtt{i}\}$ and since $u(q)(1-q) < 0$ it follows that $\alpha_1(\mathtt{i})^{u(q)(1-q)} \leq \min\{c_\mathtt{i}^{u(q)(1-q)}, d_\mathtt{i}^{u(q)(1-q)}\}$. Then

$$\sum_{\mathtt{i}\in\mathcal{I}^k} \phi^{u(q)}(T_\mathtt{i})^{1-q} p_\mathtt{i}^q \leq \min\left\{\sum_{\mathtt{i}\in\mathcal{I}^k} c_\mathtt{i}^{u(q)(1-q)} p_\mathtt{i}^q, \sum_{\mathtt{i}\in\mathcal{I}^k} d_\mathtt{i}^{u(q)(1-q)} p_\mathtt{i}^q\right\} = P_0^*(u(q), q) = 1,$$

where we have used the fact that $c_\mathtt{i}$ and $d_\mathtt{i}$ are multiplicative in $\mathtt{i}$. Therefore, for $t = u(q)$,

$$\lim_{k\to\infty} \left(\sum_{\mathtt{i}\in\mathcal{I}^k} \phi^t(T_\mathtt{i})^{1-q} p_\mathtt{i}^q\right)^{1/k} \leq 1$$

and since the expression on the left is *increasing* in t (as $q > 1$)

$$d_q(T_1, \ldots, T_N, \mu) \geq u(q).$$

If $1 \leq u(q) < 2$ the proof follows similarly as

$$\begin{aligned}
\phi^{u(q)}(T_\mathtt{i})^{1-q} &= \left(\alpha_1(\mathtt{i})\alpha_2(\mathtt{i})^{u(q)-1}\right)^{1-q} \\
&\leq \min\left\{\left(c_\mathtt{i}^{2-u(q)}(c_\mathtt{i}\ d_\mathtt{i})^{u(q)-1}\right)^{1-q},\ \left(d_\mathtt{i}^{2-u(q)}(c_\mathtt{i}\ d_\mathtt{i})^{u(q)-1}\right)^{1-q}\right\}.
\end{aligned}$$

□

Despite this simple improvement on the lower bound, we prove that $d_q(T_1, \ldots, T_N, \mu)$ is still *not* generally equal to $u(q)$ for $q > 1$.

Theorem 2.4.5. *Let c, d be such that $c > d > 0$ and $c + d \leq 1$. Let μ be the self-affine measure defined by the probability vector $(1/2, 1/2)$ and the diagonal system consisting* *of the two maps, T_1 and T_2, defined by*

$$T_1(x,y) = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textit{and} \qquad T_2(x,y) = \begin{pmatrix} d & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1-d \\ 1-c \end{pmatrix}.$$

For $q > 1$ let $u(q)$ be defined by $P_0^*(u(q), q) = 1$, that is $u(q)$ is the unique solution of

$$c^{u(q)(1-q)} 2^{-q} + d^{u(q)(1-q)} 2^{-q} = 1.$$

Then, for all $q > 1$,

$$d_q(T_1, T_2, \mu) > u(q).$$

More precisely, for all $q > 1$,

$$d_q(T_1, T_2, \mu) \geq u(q) + \frac{2 \log\left(\frac{2(c/d)^{u(q)(q-1)/2}}{(c/d)^{u(q)(q-1)}+1}\right)}{(q-1)\log(cd)}.$$

Proof. We adapt the proof of Theorem 2.3.14, although we first show that $0 \leq u(q) \leq 1$ for $q > 1$. In this case

$$P(0,q) = 2^{-q} + 2^{-q} < \frac{1}{2} + \frac{1}{2} = 1.$$

If we now consider $P(1,q)$, then as $c + d \leq 1$ (and $q > 1$)

$$\begin{aligned} P(1,q) &= c^{1-q} 2^{-q} + d^{1-q} 2^{-q} \\ &\geq c^{1-q} 2^{-q} + (1-c)^{1-q} 2^{-q}. \end{aligned}$$

Viewing

$$c^{1-q} 2^{-q} + (1-c)^{1-q} 2^{-q}$$

as a function of $c \in [0,1]$, it is easy to see (by taking the derivative) that it is decreasing on $[0,1/2]$ and increasing on $[1/2,1]$ and therefore has a minimum at $c=1/2$. Therefore

$$\begin{aligned} P(1,q) &\geq c^{1-q}2^{-q} + (1-c)^{1-q}2^{-q} \\ &\geq 2^{-(1-q)}2^{-q} + 2^{-(1-q)}2^{-q} \\ &= 1. \end{aligned}$$

As $P(0,q) \leq 1$ and $P(1,q) \geq 1$, it follows from continuity of P that $0 \leq u(q) \leq 1$.

Now let k be odd and consider the following sum

$$\sum_{\mathrm{i}\in\mathcal{I}^k} \phi^{u(q)}(T_{\mathrm{i}})^{1-q} p_{\mathrm{i}}^q = \sum_{\mathrm{i}\in\mathcal{I}^k} \alpha_1(\mathbf{i})^{u(q)(1-q)} 2^{-kq}. \tag{2.4.6}$$

As in the proof of Theorem 2.3.14 we see that for $\mathbf{i} \in \mathcal{I}^k$ if T_1 appears i times in the composition of T_{i} and T_2 appears $k-i$ times, then, since $c > d$,

$$\alpha_1(\mathbf{i}) = c^{\max\{i,k-i\}} \times d^{\min\{i,k-i\}}$$

so (2.4.6) is equal to

$$\sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k}{i} \left(c^{k-i}d^i\right)^{u(q)(1-q)} 2^{-kq} + \sum_{i=\lceil k/2 \rceil}^{k} \binom{k}{i} \left(d^{k-i}c^i\right)^{u(q)(1-q)} 2^{-kq}.$$

We again define X_k^q and Y_k^q to be the left and right terms of this expression. Continuing with exactly the same approach as in the proof of Theorem 2.3.14 and applying Lemma 2.2.1, where in this case $x = (c/d)^{u(q)(q-1)} > 1$, we find that

$$\lim_{k\to\infty} \left(\sum_{\mathrm{i}\in\mathcal{I}^k} \phi^{u(q)}(T_{\mathrm{i}})^{1-q} p_{\mathrm{i}}^q \right)^{1/k} = \lim_{k\to\infty} (X_k^q + Y_k^q)^{1/k} = \frac{2(c/d)^{u(q)(q-1)/2}}{(c/d)^{u(q)(q-1)} + 1} =: \delta < 1.$$

Recalling that since $1-q<0$, it follows in this setting that

$$\lim_{k\to\infty}\left(\sum_{\mathtt{i}\in\mathcal{I}^k}\phi^t(T_{\mathtt{i}})^{1-q}p_{\mathtt{i}}^q\right)^{1/k}$$

is a strictly *increasing* function of t and therefore

$$d_q(T_1,T_2,\mu)>u(q)$$

as required. We can upgrade this result to get the stated quantitative lower bound by studying the definition of $d_q(T_1,T_2,\mu)$ more closely. For $k\geq 1$ and $\mathtt{i}\in\mathcal{I}^k$, the larger singular value satisfies $\alpha_1(\mathtt{i})\geq(cd)^{k/2}$ and therefore, for $\varepsilon=d_q(T_1,T_2,\mu)-u(q)>0$,

$$\begin{aligned}
\delta=\lim_{k\to\infty}\left(\sum_{\mathtt{i}\in\mathcal{I}^k}\phi^{u(q)}(T_{\mathtt{i}})^{1-q}p_{\mathtt{i}}^q\right)^{1/k} &= \lim_{k\to\infty}\left(\sum_{\mathtt{i}\in\mathcal{I}^k}\alpha_1(\mathtt{i})^{u(q)(1-q)}2^{-kq}\right)^{1/k}\\
&\geq (cd)^{\varepsilon(q-1)/2}\lim_{k\to\infty}\left(\sum_{\mathtt{i}\in\mathcal{I}^k}\alpha_1(\mathtt{i})^{d_q(T_1,T_2,\mu)(1-q)}2^{-kq}\right)^{1/k}\\
&= (cd)^{\varepsilon(q-1)/2}
\end{aligned}$$

and therefore

$$d_q(T_1,T_2,\mu)-u(q)=\varepsilon\geq\frac{2\log\delta}{(q-1)\log(cd)},$$

which proves the theorem. □

2.4.2 New closed form bounds for generalised dimensions

Despite the fact that $d_q(T_1,\ldots,T_N,\mu)$ is not given by the value predicted by Falconer-Miao [18, 42] $q>1$, we can still find upper bounds in the case when our matrices are diagonal by following the approach of Section 2.3.3. To simplify notation and aid readability we only pursue such bounds in the planar case - although we expect similar bounds to hold more generally there can be obstacles when trying to generalise results about self-affine sets to higher dimensions (see for instance [27]).

For convenience we let $\mathcal{I}$ denote the set $\{1,\ldots,N\}$. We also let t_1, t_2, s_1, s_2 be defined by the following equations:

$$\sum_{i=1}^{N} p_i^q \ c_i^{t_1(1-q)} = 1, \qquad \sum_{i=1}^{N} p_i^q \ d_i^{t_2(1-q)} = 1,$$
$$\sum_{i=1}^{N} p_i^q \left(c_i^{2-s_1}(c_i \ d_i)^{s_1-1}\right)^{1-q} = 1, \qquad \sum_{i=1}^{N} p_i^q \left(d_i^{2-s_2}(c_i \ d_i)^{s_2-1}\right)^{1-q} = 1,$$

and, as in the previous section, define $u(q)$ by $P_0^*(u(q),q) = 1$. We may assume that $u(q) < 2$, as otherwise there is nothing to prove, and we note that $u(q)$ is always equal to one of t_1, t_2, s_1, s_2. Once again we write $x^+ = \max\{x, 0\}$.

Theorem 2.4.7. *Let μ be a self-affine measure generated by a diagonal system in $\mathbb{R}^2$ and assume that $q > 1$.*

(a) If $1 \leq u(q) < 2$ then

$$d_q(T_1,\ldots,T_N,\mu) \leq \min\{U_1(q), U_2(q)\},$$

where

$$U_1(q) = s_1 + \left((2-s_1)\frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{1-q} \ d_i^{(s_1-1)(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{1-q} \ d_i^{(s_1-1)(1-q)} \log(c_i)}\right)^+$$

and

$$U_2(q) = s_2 + \left((2-s_2)\frac{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{1-q} \ c_i^{(s_2-1)(1-q)} \log(d_i/c_i)}{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{1-q} \ c_i^{(s_2-1)(1-q)} \log(d_i)}\right)^+.$$

Here

$$\frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{1-q} \ d_i^{(s_1-1)(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{1-q} \ d_i^{(s_1-1)(1-q)} \log(c_i)} < 1$$

and

$$\frac{\sum_{i\in\mathcal{I}} p_i^q \; d_i^{1-q} \; c_i^{(s_2-1)(1-q)} \log(d_i/c_i)}{\sum_{i\in\mathcal{I}} p_i^q \; d_i^{1-q} \; c_i^{(s_2-1)(1-q)} \log(d_i)} < 1,$$

which we emphasise as it ensures that this is a strictly better bound than $d_q(T_1, \dots, T_N, \mu) \leq 2$.

(b) *If* $0 \leq u(q) < 1$ *let*

$$V_1(q) = t_1 + \left(t_1 \frac{\sum_{i\in\mathcal{I}} p_i^q \; c_i^{t_1(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \; c_i^{t_1(1-q)} \log(d_i)} \right)^+$$

and

$$V_2(q) = t_2 + \left(t_2 \frac{\sum_{i\in\mathcal{I}} p_i^q \; d_i^{t_2(1-q)} \log(d_i/c_i)}{\sum_{i\in\mathcal{I}} p_i^q \; d_i^{t_2(1-q)} \log(c_i)} \right)^+ .$$

(i) *If* $\min\{V_1(q), V_2(q)\} \leq 1$ *then*

$$d_q(T_1, \dots, T_N, \mu) \leq \min\{V_1(q), V_2(q)\}.$$

(ii) *If* $\min\{V_1(q), V_2(q)\} > 1$ *then*

$$d_q(T_1, \dots, T_N, \mu) \leq \min\{W_1(q), W_2(q)\}$$

where

$$W_1(q) = t_1 + \max\{A(q), C(q)\}^+$$

and

$$W_2(q) = t_2 + \max\{B(q), D(q)\}^+,$$

and where

$$
\begin{aligned}
A(q) &= (1-t_1)\frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(d_i/c_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(d_i)}, \\
B(q) &= (1-t_2)\frac{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{t_2(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{t_2(1-q)} \log(c_i)}, \\
C(q) &= \frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(c_i)}, \\
D(q) &= \frac{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{t_2(1-q)} \log(d_i/c_i)}{\sum_{i\in\mathcal{I}} p_i^q \ d_i^{t_2(1-q)} \log(d_i)}.
\end{aligned}
$$

Proof. The proof follows the strategy of the proof of Theorem 2.3.24 and so we suppress some common details. Let $\{\theta_i\}_{i\in\mathcal{I}}$ denote an arbitrary probability vector and, for each $k \in \mathbb{N}$, define $n(k) \in \mathbb{N}$ by

$$n(k) = \sum_{i\in\mathcal{I}} \lfloor \theta_i k \rfloor.$$

Recall that $k - |\mathcal{I}| \leq n(k) \leq k$. We again consider the $n(k)$th iteration of $\mathcal{I}$ and define

$$\mathcal{J}_k = \left\{ \mathrm{j} = (j_1, \dots, j_{n(k)}) \in \mathcal{I}^{n(k)} : \#\{m : j_m = i\} = \lfloor \theta_i k \rfloor \text{ for each } i \in \mathcal{I} \right\},$$

with the size $\mathcal{J}_k$ given by

$$|\mathcal{J}_k| = \frac{n(k)!}{\prod_{i\in\mathcal{I}} \lfloor \theta_i k \rfloor !}.$$

We also define numbers c, d and p by

$$c = \prod_{i\in\mathcal{I}} c_i^{\lfloor \theta_i k \rfloor}, \qquad d = \prod_{i\in\mathcal{I}} d_i^{\lfloor \theta_i k \rfloor}, \qquad p = \prod_{i\in\mathcal{I}} p_i^{\lfloor \theta_i k \rfloor}.$$

(a) Firstly we assume $1 \leq u(q) < 2$, so in this case $u(q)$ is given by either s_1 and s_2, which are defined above. We also assume that $\prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i}$. We know from the proof of Theorem 2.3.24 that this condition implies that $c > d$ for k sufficiently large.

We then have that for all $\mathtt{i} \in \mathcal{J}_k$ and $s > 0$ that

$$\phi^s(T_{\mathtt{i}})^{1-q} p_{\mathtt{i}}^q = (c\ d^{s-1})^{1-q} p^q = p^q\ c^{1-q}\ d^{(s-1)(1-q)},$$

which by definition of p, c and d we may write as

$$\phi^s(T_{\mathtt{i}})^{1-q} p_{\mathtt{i}}^q = \prod_{i \in \mathcal{I}} \left(p_i^q\ c_i^{1-q}\ d_i^{(s-1)(1-q)} \right)^{\lfloor \theta_i k \rfloor}.$$

Using exactly the same reasoning as in the proof of Theorem 2.3.24 (simply replacing $p_i^q\ c_i^{\tau_1(q)}\ d_i^{s-\tau_1(q)}$ by $p_i^q\ c_i^{1-q}\ d_i^{(s-1)(1-q)}$) we may show that

$$\begin{aligned}
&\log\left(\left(\sum_{\mathtt{i} \in \mathcal{I}^{n(k)}} \phi^s(T_{\mathtt{i}})^{1-q} p_{\mathtt{i}}^q\right)^{1/n(k)}\right) \\
&\geq \log\left(\frac{k - |\mathcal{I}|}{k}\right) - \frac{1}{k - |\mathcal{I}|} \sum_{i \in \mathcal{I}} \log \theta_i k + \sum_{i \in \mathcal{I}} \theta_i \log\left(\frac{p_i^q\ c_i^{1-q}\ d_i^{(s-1)(1-q)}}{\theta_i}\right) \\
&\to \sum_{i \in \mathcal{I}} \theta_i \log\left(\frac{p_i^q\ c_i^{1-q}\ d_i^{(s-1)(1-q)}}{\theta_i}\right)
\end{aligned}$$

as $k \to \infty$. If this is greater than or equal to 0

$$\lim_{k \to \infty} \left(\sum_{\mathtt{i} \in \mathcal{I}^k} \phi^s(T_{\mathtt{i}})^{1-q} p_{\mathtt{i}}^q\right)^{1/k} \geq 1$$

and therefore

$$d_q(T_1, \ldots, T_N, \mu) \leq s.$$

This follows because when $q > 1$ the above limit is a strictly increasing function of s (as opposed to when $0 < q < 1$, when it is a strictly decreasing function of s). As before we can use a very similar argument when $\prod_{i \in \mathcal{I}} c_i^{\theta_i} \leq \prod_{i \in \mathcal{I}} d_i^{\theta_i}$. Combining these cases we find

$$d_q(T_1,\ldots,T_N,\mu) \le \inf \Bigg\{ s : \text{there exists a probability vector } \{\theta_i\}_{i\in\mathcal{I}} \text{ such that either}$$

$$(1)\ \prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i} \text{ and} \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ c_i^{1-q}\ d_i^{(s-1)(1-q)}}{\theta_i}\right) \ge 0$$

$$\text{or } (2)\ \prod_{i\in\mathcal{I}} c_i^{\theta_i} < \prod_{i\in\mathcal{I}} d_i^{\theta_i} \text{ and} \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ d_i^{1-q}\ c_i^{(s-1)(1-q)}}{\theta_i}\right) \ge 0$$

$$\text{or } (3)\ \prod_{i\in\mathcal{I}} c_i^{\theta_i} = \prod_{i\in\mathcal{I}} d_i^{\theta_i} \text{ and both} \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ c_i^{1-q}\ d_i^{(s-1)(1-q)}}{\theta_i}\right) \ge 0$$

$$\text{and} \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ d_i^{1-q}\ c_i^{(s-1)(1-q)}}{\theta_i}\right) \ge 0 \Bigg\}. \tag{2.4.8}$$

Once again, we have the freedom to choose a probability vector. Natural choices here would be to take either $\{p_i^q \left(c_i^{2-s_1}(c_i\ d_i)^{s_1-1}\right)^{1-q}\}_{i\in\mathcal{I}}$ or $\{p_i^q \left(d_i^{2-s_2}(c_i\ d_i)^{s_2-1}\right)^{1-q}\}_{i\in\mathcal{I}}$, which by definition of s_1 and s_2 are indeed probability vectors. Recall that $u(q)$ is given by either s_1 or s_2. Choose

$$\{\theta_i\}_{i\in\mathcal{I}} = \left\{p_i^q \left(c_i^{2-s_1}(c_i\ d_i)^{s_1-1}\right)^{1-q}\right\}_{i\in\mathcal{I}} = \left\{p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)}\right\}_{i\in\mathcal{I}}.$$

We also replace s in (2.4.8) by $s_1+\varepsilon$, where $\varepsilon \ge 0$ is small enough so that $1 < s_1+\varepsilon < 2$ (this clearly does not affect any of the above calculations). We want to investigate how small we can choose ε. We again require two conditions to hold, namely

$$\sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ c_i^{1-q}\ d_i^{(s_1+\varepsilon-1)(1-q)}}{\theta_i}\right) \ge 0 \tag{2.4.9}$$

and

$$\sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q\ d_i^{1-q}\ c_i^{(s_1+\varepsilon-1)(1-q)}}{\theta_i}\right) \ge 0 \tag{2.4.10}$$

(this is because again we do not know if $\prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i}$ or $\prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i}$). The condition (2.4.9) holds trivially since

$$\begin{aligned}&\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log\left(\frac{p_i^q\ c_i^{1-q}\ d_i^{(s_1+\varepsilon-1)(1-q)}}{p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)}}\right)\\ =&\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log\left(d_i^{\varepsilon(1-q)}\right) \geq 0.\end{aligned}$$

For (2.4.10) to hold, we require

$$\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log\left(\frac{p_i^q\ d_i^{1-q}\ c_i^{(s_1+\varepsilon-1)(1-q)}}{p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)}}\right) \geq 0,$$

which upon rearranging is equivalent to

$$\varepsilon \geq (2-s_1)\frac{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(c_i)}. \tag{2.4.11}$$

This implies

$$d_q(T_1,\ldots,T_N,\mu) \leq s_1 + \left((2-s_1)\frac{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(c_i)}\right)^+ = U_1(q).$$

If the bound for ε in (2.4.11) is negative then we take $\varepsilon = 0$, which is why the $^+$ appears. Finally

$$\frac{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(c_i)} = 1 - \frac{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(d_i)}{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{1-q}\ d_i^{(s_1-1)(1-q)} \log(c_i)} < 1,$$

so our upper bound is an improvement on

$$d_q(T_1,\ldots,T_N,\mu) \leq 2.$$

The other upper bound $d_q(T_1, \ldots, T_N, \mu) \leq U_2(q)$ is proved similarly and relies on the other natural choice of $\{\theta_i\}$.

(b) We shall now assume that $0 \leq u(q) < 1$, so here $u(q)$ is given by either t_1 or t_2, defined above. Looking again at the $n(k)$th iteration of $\mathcal{I}$ and first supposing that $\prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i}$, then for all $\mathbf{i} \in \mathcal{J}_k$ and $s \in [0,1]$

$$\phi^s(T_{\mathbf{i}})^{1-q} p_{\mathbf{i}}^q = p^q \; c^{s(1-q)}.$$

We use exactly the same reasoning as above and find that in this case

$$\begin{aligned} d_q(T_1, \ldots, T_N, \mu) \leq \inf \Bigg\{ s \in [0,1] : &\text{ there exists a probability vector } \{\theta_i\}_{i\in\mathcal{I}} \text{ such that either} \\ &(1) \; \prod_{i\in\mathcal{I}} c_i^{\theta_i} > \prod_{i\in\mathcal{I}} d_i^{\theta_i} \;\text{ and } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q \; c_i^{s(1-q)}}{\theta_i}\right) \geq 0 \\ \text{or } &(2) \; \prod_{i\in\mathcal{I}} c_i^{\theta_i} < \prod_{i\in\mathcal{I}} d_i^{\theta_i} \;\text{ and } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q \; d_i^{s(1-q)}}{\theta_i}\right) \geq 0 \\ \text{or } &(3) \; \prod_{i\in\mathcal{I}} c_i^{\theta_i} = \prod_{i\in\mathcal{I}} d_i^{\theta_i} \;\text{ and both } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q \; c_i^{s(1-q)}}{\theta_i}\right) \geq 0 \\ &\text{and } \sum_{i\in\mathcal{I}} \theta_i \log\left(\frac{p_i^q \; d_i^{s(1-q)}}{\theta_i}\right) \geq 0 \Bigg\}. \end{aligned}$$

There is a complication here as we require $s \leq 1$ because we assume the singular value function takes the form α_1^s. This is what leads to the awkward extra case in the $u(q) < 1$ setting.

Again, there are two natural choices for probability vector $\{\theta_i\}$, the first of which is

$$\{\theta_i\}_{i\in\mathcal{I}} = \left\{p_i^q \; c_i^{t_1(1-q)}\right\}_{i\in\mathcal{I}}.$$

We replace s by $t_1 + \varepsilon$ in the above, where $\varepsilon \geq 0$. Once again we would like to see how small it is possible to take ε. There are two possibilities: when ε can be taken sufficiently small so that $t_1+\varepsilon<1$ and when $1\leq t_1+\varepsilon<2$ (this will affect which form of the singular value function we can use).

(i) Firstly suppose we can take ε sufficiently small so that $t_1+\varepsilon<1$. We require two conditions to hold, the first of which is trivial since

$$\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log\left(\frac{p_i^q \ c_i^{(t_1+\varepsilon)(1-q)}}{p_i^q \ c_i^{t_1(1-q)}}\right) = \sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(c_i^{\varepsilon(1-q)}) \geq 0.$$

For the second condition to hold, we require

$$\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log\left(\frac{p_i^q \ d_i^{(t_1+\varepsilon)(1-q)}}{p_i^q \ c_i^{t_1(1-q)}}\right) \geq 0,$$

which is equivalent to

$$\varepsilon \geq t_1 \frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(d_i)}.$$

This implies that

$$d_q(T_1,\dots,T_N,\mu) \leq t_1 + \left(t_1 \frac{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(d_i)}\right)^+.$$

(ii) Now suppose we cannot take ε sufficiently small so that $t_1+\varepsilon<1$, so we instead have to look at what happens when $1\leq t_1+\varepsilon<2$. In this case we will still be using the same choice of probability vector but we will be using the form of the singular value function in the range $[1,2]$, that is $\alpha_1\alpha_2^{s-1}$, and we refer to the general upper bound in the case $1\leq u(q)<2$ given above.

As usual we require two conditions to hold simultaneously, but this time neither condition is trivial. We require

$$\sum_{i\in\mathcal{I}} p_i^q\ c_i^{t_1(1-q)} \log\left(\frac{p_i^q\ c_i^{1-q}\ d_i^{(t_1+\varepsilon-1)(1-q)}}{p_i^q\ c_i^{t_1(1-q)}}\right) \geq 0,$$

which is equivalent to $\varepsilon \geq A(q)$, where

$$A(q) = (1-t_1)\frac{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{t_1(1-q)} \log(d_i/c_i)}{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{t_1(1-q)} \log(d_i)}.$$

We also require

$$\sum_{i\in\mathcal{I}} p_i^q\ c_i^{t_1(1-q)} \log\left(\frac{p_i^q\ d_i^{1-q}\ c_i^{(t_1+\varepsilon-1)(1-q)}}{p_i^q\ c_i^{t_1(1-q)}}\right) \geq 0,$$

which is equivalent to $\varepsilon \geq C(q)$, where

$$C(q) = \frac{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{t_1(1-q)} \log(c_i/d_i)}{\sum_{i\in\mathcal{I}} p_i^q\ c_i^{t_1(1-q)} \log(c_i)}.$$

Thus we may conclude in this instance that

$$d_q(T_1,\ldots,T_N,\mu) \leq t_1 + \max\{A(q),C(q)\}^+ = W_1(q).$$

The other upper bound, $W_2(q)$, can be derived similarly. $\square$

As a corollary to the above, we present simple conditions that ensure $d_q(T_1,\ldots,T_N,\mu) = u(q)$, that is, for the theorem of Falconer-Miao to hold when $q > 1$.

Corollary 2.4.12. *Consider the diagonal system of Theorem 2.4.7 and $q > 1$. First suppose that $1 < u(q) \leq 2$. If $u(q) = s_1$ and*

$$\sum_{i \in \mathcal{I}} p_i^q \ c_i^{1-q} \ d_i^{(s_1-1)(1-q)} \log(c_i/d_i) \geq 0,$$

then $d_q(T_1, \ldots, T_N, \mu) = u(q) = s_1$. If $u(q) = s_2$ and

$$\sum_{i \in \mathcal{I}} p_i^q \ d_i^{1-q} \ c_i^{(s_2-1)(1-q)} \log(d_i/c_i) \geq 0,$$

then $d_q(T_1, \ldots, T_N, \mu) = u(q) = s_2$. Secondly, suppose that $0 < u(q) \leq 1$. If $u(q) = t_1$ and

$$\sum_{i \in \mathcal{I}} p_i^q \ c_i^{t_1(1-q)} \log(c_i/d_i) \geq 0,$$

then $d_q(T_1, \ldots, T_N, \mu) = u(q) = t_1$. If $u(q) = t_2$ and

$$\sum_{i \in \mathcal{I}} p_i^q \ d_i^{t_2(1-q)} \log(d_i/c_i) \geq 0,$$

then $d_q(T_1, \ldots, T_N, \mu) = u(q) = t_2$.

In particular, if $c_i \geq d_i$ for all $i \in \mathcal{I}$ or $c_i \leq d_i$ for all $i \in \mathcal{I}$, then $d_q(T_1, \ldots, T_N, \mu) = u(q)$.

Proof. This follows from Theorem 2.4.7, noting in each instance that if one of these conditions holds then we may choose $\varepsilon = 0$. □

2.4.3 An example

Here we present an example of a diagonal system with three maps to which Corollary 2.4.12 can be applied. We take $p_1 = 4/5, p_2 = 1/10, p_3 = 1/10$ as our probability vector and define three maps by choosing $c_1 = 2/5, c_2 = 3/10, c_3 = 3/10$ and $d_1 = 3/10, d_2 =$

$2/5, d_3 = 3/10$. For $q \in [0,5]$ we have $0 < u(q) \leq 1$ and

$$\sum_{i \in \mathcal{I}} p_i^q \; c_i^{t_1(1-q)} \log(c_i/d_i) \geq 0,$$

which means the first condition from Corollary 2.4.12 is satisfied. Therefore $d_q(T_1, T_2, T_3, \mu) = u(q) = t_1$ for $q \in [0,5]$ by Corollary 2.4.12, see Figure 2.3.

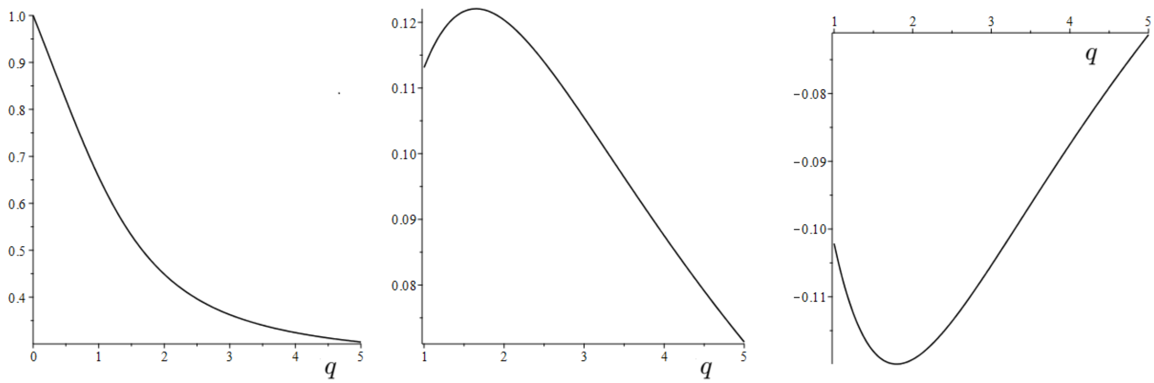

Figure 2.3: Left: plot of $d_q(T_1, T_2, T_3, \mu) = u(q)$. Previously this formula was only known for $0 < q < 1$, see [18, 42]. Middle: plot of the first condition from Corollary 2.4.12, which is satisfied for the whole range of q. Right: plot of the second condition from Corollary 2.4.12, which is not satisfied.

Observe that the value at $q = 0$ gives the affinity dimension of the support of our measure, which in this case is 1. Also recall that by Falconer's result [14, Theorem 6.2], if the translation vectors are randomised then the generalised q-dimensions of μ are given by $d_q(T_1, T_2, T_3, \mu)$ for $1 < q \leq 2$ almost surely, provided the norms of the matrices are strictly less than 1/2, see Figure 2.4.

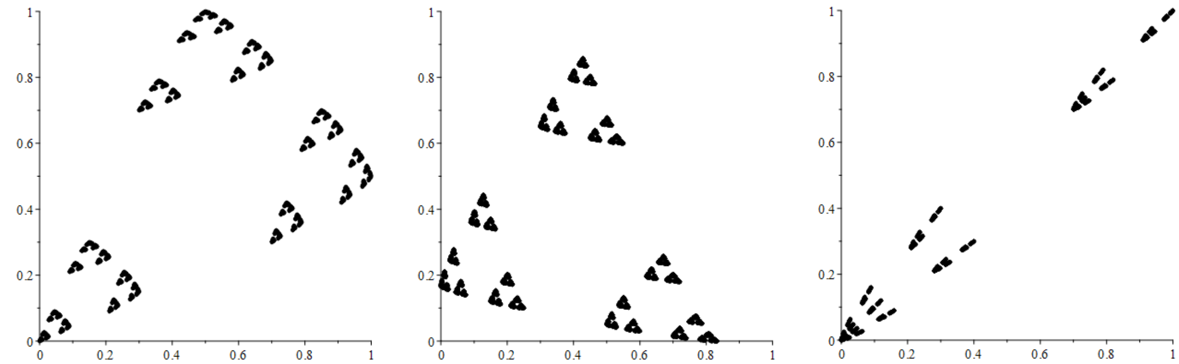

Figure 2.4: Three self-affine sets generated by the collection of matrices given in Example 2.4.3 with translations which have been chosen differently. Therefore the self-affine measures from Example 2.4.3 they support almost surely have generalised q-dimensions given by $u(q) = d_q(T_1, T_2, T_3, \mu)$ for $1 < q \leq 2$.

Chapter 3

L^q-spectra of measures on nonlinear attractors

3.1 Background

The study of fractals generated by iterated function systems (IFSs) consisting of nonlinear maps, which can often be identified with repellers of corresponding dynamical systems, has a rich history, with some particularly notable progress over the past thirty years. In 1994 Falconer [12] calculated the box dimension of certain mixing repellers for non-conformal mappings. To do this he applied techniques from thermodynamic formalism, in particular developing a subadditive version of the theory and also a "bounded distortion" principle. Further work on nonlinear IFSs was done by Hu who in 1996 calculated the box and Hausdorff dimensions of invariant sets of expanding C^2 maps [30]. More recent work includes that of Cao, Pesin and Zhao [9] as well as that of Feng and Simon [22]. Cao, Pesin and Zao studied the Hausdorff dimension of non-conformal repellers corresponding to $C^{1+\alpha}$ maps. By studying certain subadditive and superadditive pressures they were able to obtain bounds for the Hausdorff dimension of repellers. Feng and Simon proved that the upper box dimension of the attractor of any C^1 IFS in $\mathbb{R}^n$ is bounded above by its singularity dimension.

Other notable work in this area was done in 2007 by Manning and Simon [39] who investigated the subadditive pressure of nonlinear maps developed by Falconer and studied cases where bounded distortion does not hold. The work of Falconer as well as that of Manning and Simon and also Miao [18] was generalised by Bárány [1] who used the subadditive pressure to calculate the Hausdorff dimension of fractals generated by IFSs whose maps have triangular Jacobians. Other authors to have studied IFSs generated by triangular mappings include Kolossváry and Simon [36]. In particular they looked at a family of planar self-affine carpets with overlaps generated by lower triangular matrices and asked whether a dimension drop occurs.

In terms of multifractal analysis Falconer studied the L^q-spectrum of self-affine measures [14] and almost self-affine measures [15]. In the case of self-affine measures he was able to establish a generic formula in the region $1 < q \leq 2$ in terms of a subadditive pressure expression. Barral and Feng [5] then generalised this in certain cases to calculate the L^q-spectrum for a wider range of q and were also able to verify the multifractal formalism in some cases. For results on the L^q-spectrum of measures on self-affine carpets, see King [34], Olsen [45], Jordan and Rams [32], Feng and Wang [23] and Fraser [25].

In this chapter we calculate the L^q-spectra of Bernoulli measures in the plane supported on sets generated by IFSs consisting of $C^{1+\alpha}$ maps whose Jacobian matrices are lower triangular. Our approach is based on setting up certain 'almost-additive' pressure functionals. As a corollary we calculate the box dimension of the supports of these measures. Our results on L^q-dimensions are new, even in the (non-diagonal) self-affine case.

3.2 Nonlinear attractors and measures

We begin by introducing further definitions, in particular *nonlinear attractors* and *nonlinear measures* which have a particular meaning in this chapter as shorthand for the types of non-conformal attractors and measures we study.

Definition 3.2.1 (Nonlinear attractor)**.** Let $\mathcal{I}$ be a finite index set with $|\mathcal{I}| \geq 2$ and let $\{S_i\}_{i\in\mathcal{I}}$ be an IFS consisting of contractions on $[0,1]^2$. Suppose also that each $S_i : [0,1]^2 \to [0,1]^2$ is of the form $S_i(a_1, a_2) = (f_i(a_1), g_i(a_1, a_2))$, where the f_i and g_i are $C^{1+\alpha}$ contractions $(0 < \alpha \leq 1)$ on $[0,1]$ and $[0,1]^2$ respectively, that is their derivatives satisfy Hölder conditions of exponent α. (We use one-sided derivatives on the boundary of $[0,1]^2$.) By Hutchinson's theorem (Theorem 1.2.2) there is a unique non-empty, compact set F satisfying

$$F = \bigcup_{i\in\mathcal{I}} S_i(F),$$

which for the purposes of this paper we call the *nonlinear attractor* associated to $\{S_i\}_{i\in\mathcal{I}}$.

The observant reader may notice that in our definition of nonlinear attractor the maps in the IFS can be linear. Our choice of name is to emphasise that our focus is on the situation when these maps are nonlinear, as this is where we obtain new results.

We are interested in the natural Bernoulli measures (Definition (1.7.2)) supported on nonlinear attractors F.

Definition 3.2.2 (Nonlinear measure)**.** Let F be a nonlinear attractor given by $\{S_i\}_{i\in\mathcal{I}}$ on $[0,1]^2$, and let $\{p_i\}_{i\in\mathcal{I}}$ be a probability vector with each $p_i \in (0,1)$. Then there is a unique Borel probability measure μ supported on F which satisfies (as in Definition

1.4.1)

$$\mu = \sum_{i \in \mathcal{I}} p_i \; \mu \circ S_i^{-1},$$

which we call the *nonlinear measure* associated to $\{S_i\}_{i \in \mathcal{I}}$ and $\{p_i\}_{i \in \mathcal{I}}$.

Our aim is to calculate the L^q-spectra (Definition 1.6.3) of these measures. To calculate this for nonlinear measures we require an appropriate separation condition, which is needed to establish the lower bound. In particular we shall assume our IFS satisfies the rectangular open set condition (Definition 2.3.3). Although we expect our results to hold for the more general open set condition we choose to use $(0, 1)^2$ as it is more convenient in our setting.

Fraser [25] calculated the L^q-spectrum $\tau_\mu(q)$ of a class of *self-affine* measures in the plane. We broadly follow his approach although there are several challenges which arise due to the nonlinearity, as well as the maps giving rise to non-diagonal Jacobians. In particular we require several additional technical results (Lemmas 3.4.9, 3.4.11 and 3.4.13) in order to define an appropriate pressure function. It is also much more difficult to compare moment sums of the measure to moment sums of the measure projected onto the x-axis. In [25, Lemma 7.2] Fraser was able to establish this in relatively few lines whereas the analogue our setting (Lemma 3.5.5) has a far more involved and challenging proof.

Our main result Theorem 3.4.24 requires some more assumptions and technical details, in particular that the $\{S_i\}_{i \in \mathcal{I}}$ contract more in the vertical direction than in the horizontal direction. The theorem is stated fully in Section 3.4 but the essence of it is captured in the following version.

Theorem 3.2.3. *Let μ be a nonlinear measure which satisfies a natural domination condition and the ROSC and let $q \geq 0$. Then there exists a function $\gamma : [0, \infty) \to$*

$\mathbb{R}$, defined in terms of the probability vector $\{p_i\}_{i\in\mathcal{I}}$, the singular values of Jacobian matrices of iterates of the $\{S_i\}_{i\in\mathcal{I}}$ and the L^q-spectrum of the projection of μ onto the x-axis, such that

$$\tau_\mu(q) = \gamma(q).$$

We set up the pressure formalism that enables us to define γ in Section 3.4 and prove the theorem in Section 3.5. A simple corollary is that if μ satisfies the ROSC then the box dimension of the support of μ is given by $\gamma(0)$.

3.3 An Example

Here we provide an example of a nonlinear IFS and corresponding nonlinear attractor generated by three maps. The maps are

$$\begin{aligned}
S_1(x,y) &= \left(\frac{3x}{5} + \frac{3x^2}{40},\ \frac{x^2}{12} + \frac{y}{6}\right),\\
S_2(x,y) &= \left(\frac{4x}{5} - \frac{4x^3}{30} + \frac{1}{3},\ \frac{x^2}{10} + \frac{y}{4} + \frac{17}{50}\right),\\
S_3(x,y) &= \left(\frac{3x}{5},\ \frac{x^2}{10} + \frac{y}{5} + \frac{y^3}{9} + \frac{26}{45}\right).
\end{aligned}$$

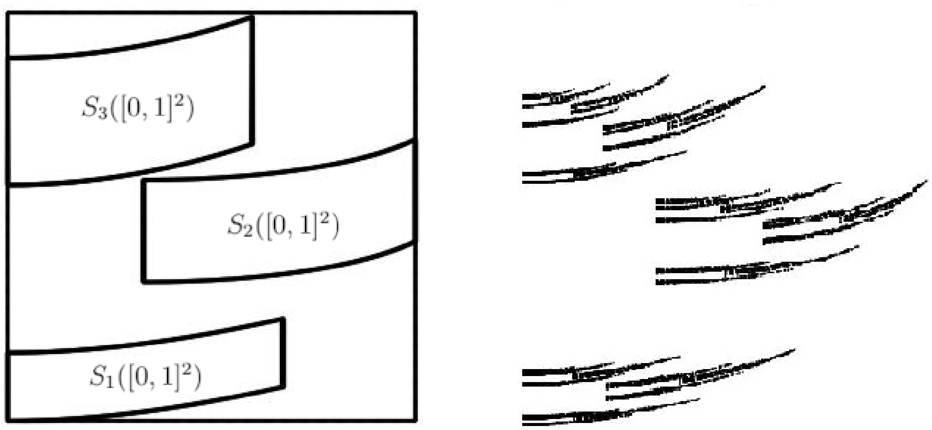

Figure 3.1: The image of the unit square $[0, 1]^2$ under the maps S_1, S_2 and S_3 is shown on the left; the IFS satisfies the ROSC (Definition 2.3.3). On the right is the corresponding nonlinear attractor.

These maps satisfy the conditions for Theorem 3.2.3 and Theorem 3.4.24. The ROSC (Definition 2.3.3) and domination condition (Definition 3.4.1, stated formally in Section 3.4) are easy to check. Indeed using Maple software gives, with $f_{i,x}$ and $g_{i,y}$ as at the start of Section 3.4,

$$\inf_{\mathbf{a}\in[0,1]^2} |f_{1,x}(\mathbf{a})| = 3/5 > \sup_{\mathbf{a}\in[0,1]^2} |g_{1,y}(\mathbf{a})| = 1/6 \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{1,y}(\mathbf{a})| = 1/6,$$

$$\inf_{\mathbf{a}\in[0,1]^2} |f_{2,x}(\mathbf{a})| = 2/5 > \sup_{\mathbf{a}\in[0,1]^2} |g_{2,y}(\mathbf{a})| = 1/4 \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{2,y}(\mathbf{a})| = 1/4,$$

$$\inf_{\mathbf{a}\in[0,1]^2} |f_{3,x}(\mathbf{a})| = 3/5 > \sup_{\mathbf{a}\in[0,1]^2} |g_{3,y}(\mathbf{a})| = 8/15 \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{3,y}(\mathbf{a})| = 1/5,$$

with $d = 1/6$, say. Thus any nonlinear measures supported on the attractor of this IFS would fall under the class studied. This example is displayed visually in Figure 3.1.

3.4 A singular value function and pressure

In Chapter 2 we saw that in [25] Fraser introduced a *q-modified singular value function*. As he was dealing with self-affine measures he needed to consider the singular values of the linear part of each affine map in the IFS. In our nonlinear setting we shall instead study singular values of Jacobian matrices.

Let $\{S_i\}_{i\in\mathcal{I}}$ be an iterated function system of the form in Definition 3.2.1. For $\mathbf{a} = (a_1, a_2) \in [0,1]^2$ and $i \in \mathcal{I}$ we denote the derivative of S_i by $D_{\mathbf{a}}S_i$. As each S_i is of the form $S_i(a_1, a_2) = (f_i(a_1), g_i(a_1, a_2))$ the Jacobian matrix of $D_{\mathbf{a}}S_i$ is a lower triangular matrix. To simplify notation we will write $S_i(\mathbf{a}) = (f_i(\mathbf{a}), g_i(\mathbf{a}))$, where $\mathbf{a} = (a_1, a_2)$, even though f does not depend on a_2. If we now write f_x for the derivative of f and g_x and g_y for the partial derivatives of g then for $i \in \mathcal{I}$

$$D_{\mathbf{a}}S_i = \begin{pmatrix} f_{i,x}(\mathbf{a}) & 0 \\ g_{i,x}(\mathbf{a}) & g_{i,y}(\mathbf{a}) \end{pmatrix}.$$

From now on we assume that the IFS satisfies the following *domination condition*, which is our key technical assumption.

Definition 3.4.1 (Domination Condition)**.** We say the IFS $\{S_i\}_{i\in\mathcal{I}}$ satisfies the *domination condition* if for each map S_i the following inequalities on the derivatives hold:

$$\inf_{\mathbf{a}\in[0,1]^2} |f_{i,x}(\mathbf{a})| > \sup_{\mathbf{a}\in[0,1]^2} |g_{i,y}(\mathbf{a})| \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{i,y}(\mathbf{a})| \geq d, \tag{3.4.2}$$

where $d > 0$.

Intuitively, the domination condition can be thought of as meaning that each map in the IFS contracts more in the vertical direction. Let

$$\eta := \sup_{i\in\mathcal{I},\mathbf{a},\mathbf{b}\in[0,1]^2} \left\{ \frac{|g_{i,y}(\mathbf{a})|}{|f_{i,x}(\mathbf{b})|} \right\} < 1 \tag{3.4.3}$$

using (3.4.2). In the obvious way we will say that μ and F satisfy the domination condition if their defining IFS does.

There is no requirement on $g_{i,x}$ to be non-zero; in particular since this allows $g_{i,x}(\mathbf{a}) = 0$ for all $\mathbf{a} \in [0,1]^2$ the class of measures we study includes self-affine measures supported on Bedford-McMullen carpets, as well as measures supported on attractors of nonlinear "diagonal" IFSs.

We write $0 < c < 1$ for the maximum contraction ratio of the S_i so in particular

$$|S_{i_1 \cdots i_k}(\mathbf{a}) - S_{i_1 \cdots i_k}(\mathbf{b})| \leq c^k |\mathbf{a} - \mathbf{b}| \qquad ((i_1, \ldots, i_k) \in \mathcal{I}^k,\ \mathbf{a}, \mathbf{b} \in [0,1]^2).$$

By the chain rule the Jacobian of the composed maps S_{i} must be lower triangular, so let $f_{\mathrm{i},x}(\mathbf{a}), g_{\mathrm{i},x}(\mathbf{a}), g_{\mathrm{i},y}(\mathbf{a})$ denote the entries of $D_{\mathbf{a}} S_{\mathrm{i}}$, that is

$$D_{\mathbf{a}} S_{\mathrm{i}} = \begin{pmatrix} f_{\mathrm{i},x}(\mathbf{a}) & 0 \\ g_{\mathrm{i},x}(\mathbf{a}) & g_{\mathrm{i},y}(\mathbf{a}) \end{pmatrix}. \tag{3.4.4}$$

We will show that the domination condition implies that these matrices satisfy a *bounded distortion* property which will be key in calculating the L^q-spectra.

Using the chain rule the diagonal entries of (3.4.4) can be written in terms of derivatives of the individual f_i and g_i as follows.

$$f_{\mathrm{i},x}(\mathbf{a}) = \prod_{j=1}^{k} f_{i_j,x}(S_{i_{j+1} \cdots i_k} \mathbf{a}), \quad g_{\mathrm{i},y}(\mathbf{a}) = \prod_{j=1}^{k} g_{i_j,y}(S_{i_{j+1} \cdots i_k} \mathbf{a}). \tag{3.4.5}$$

(Here and elsewhere we make the natural convention that $S_{i_{k+1}} S_{i_k}$ is the identity.) From

(3.4.3) and using these expansions,

$$\frac{|g_{\mathfrak{i},y}(\mathbf{a})|}{|f_{\mathfrak{i},x}(\mathbf{b})|} \;\le\; \eta^k \tag{3.4.6}$$

for all $\mathfrak{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ and all $\mathbf{a}, \mathbf{b} \in [0,1]^2$. For the bottom left term direct expansion or induction gives

$$g_{\mathfrak{i},x}(\mathbf{a}) = \sum_{j=1}^{k} G_j(\mathbf{a}) \tag{3.4.7}$$

where, using the chain rule,

$$\begin{aligned} G_j(\mathbf{a}) &= g_{i_1,y}(S_{i_2\cdots i_k}\mathbf{a})\cdots g_{i_{j-1},y}(S_{i_j\cdots i_k}\mathbf{a})g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a}) \\ &\qquad\qquad \times f_{i_{j+1},x}(S_{i_{j+2}\cdots i_k}\mathbf{a})\cdots f_{i_{k-1},x}(S_{i_k}\mathbf{a})f_{i_k,x}(\mathbf{a}) \\ &= \left(\prod_{l=1}^{j-1} g_{i_l,y}(S_{i_{l+1}\cdots i_k}\mathbf{a})\right) g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a}) \left(\prod_{l=j+1}^{k} f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{a})\right) \qquad (3.4.8) \\ &= g_{i_1\cdots i_{j-1},y}(S_{i_j\cdots i_k}\mathbf{a})g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})f_{i_{j+1}\cdots i_k,x}(S_{i_{j+2}\cdots i_k}\mathbf{a}). \end{aligned}$$

The next two lemmas obtain estimates on the entries of (3.4.4) that are uniform in $\mathfrak{i}$ and $\mathbf{a}$. The first lemma establishes a bounded distortion property for diagonal entries of Jacobian matrices.

Lemma 3.4.9. *There exists a constant $R > 0$ such that for all $\mathfrak{i} \in \mathcal{I}^*$ and all $\mathbf{a}, \mathbf{b} \in [0,1]^2$,*

$$R^{-1} \;\le\; \frac{|f_{\mathfrak{i},x}(\mathbf{a})|}{|f_{\mathfrak{i},x}(\mathbf{b})|},\; \frac{|g_{\mathfrak{i},y}(\mathbf{a})|}{|g_{\mathfrak{i},y}(\mathbf{b})|} \;\le\; R. \tag{3.4.10}$$

Proof. As each f_{i_j} is a $C^{1+\alpha}$ map there is a number B such that

$$|f_{i,x}(\mathbf{a}') - f_{i,x}(\mathbf{b}')| \le B|\mathbf{a}' - \mathbf{b}'|^{\alpha}$$

for all $i \in \mathcal{I}$ and all $\mathbf{a}', \mathbf{b}' \in [0,1]^2$. For $\mathtt{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ and $\mathbf{a}, \mathbf{b} \in [0,1]^2$, identity (3.4.5) gives

$$\begin{aligned}
\frac{|f_{\mathtt{i},x}(\mathbf{a})|}{|f_{\mathtt{i},x}(\mathbf{b})|} &= \prod_{j=1}^{k} \frac{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|} \\
&= \prod_{j=1}^{k} \left(1 + \frac{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})| - |f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}\right) \\
&\leq \prod_{j=1}^{k} \left(1 + \frac{\big||f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})| - |f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|\big|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}\right) \\
&\leq \prod_{j=1}^{k} \left(1 + \frac{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a}) - f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}\right) \\
&\leq \prod_{j=1}^{k} \left(1 + \frac{B|S_{i_{j+1}\cdots i_k}\mathbf{a} - S_{i_{j+1}\cdots i_k}\mathbf{b}|^{\alpha}}{d}\right) \\
&\leq \prod_{j=1}^{k} \left(1 + \frac{B\, c^{(k-j)\alpha}\, |\mathbf{a} - \mathbf{b}|^{\alpha}}{d}\right) \\
&\leq \prod_{j=1}^{k} \exp\left(\frac{2^{\alpha}B}{d} c^{(k-j)\alpha}\right) \\
&\leq \exp \sum_{j=1}^{k} \left(\frac{2^{\alpha}B}{d} c^{(k-j)\alpha}\right) \\
&\leq \exp\left(\frac{2^{\alpha}B}{d(1-c^{\alpha})}\right),
\end{aligned}$$

using that $|\mathbf{a} - \mathbf{b}| \leq 2$. Setting $R = \exp(2^{\alpha}B/d(1-c^{\alpha}))$ gives (3.4.10) for $f_{\mathtt{i},x}$, with the left-hand estimate obtained by reversing the roles of $\mathbf{a}$ and $\mathbf{b}$. A similar argument using (3.4.5) applies for $g_{\mathtt{i},y}$. □

We turn to the bottom left entries $g_{\mathtt{i},x}$.

Lemma 3.4.11. *There exists $C > 0$ such that for all $\mathtt{i} \in \mathcal{I}^*$ and all $\mathbf{a}, \mathbf{b} \in [0,1]^2$*

$$\left|\frac{g_{\mathtt{i},x}(\mathbf{a})}{f_{\mathtt{i},x}(\mathbf{b})}\right| \leq C. \tag{3.4.12}$$

Proof. Let $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ and $\mathbf{a}, \mathbf{b} \in [0,1]^2$. Then for $1 \le j \le k$ identities (3.4.5) and (3.4.8) give

$$\begin{aligned}
\left|\frac{G_j(\mathbf{a})}{f_{\mathbf{i},x}(\mathbf{b})}\right| &= \left|\left(\prod_{l=1}^{j-1} \frac{g_{i_l,y}(S_{i_{l+1}\cdots i_k}\mathbf{a})}{f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{b})}\right)\frac{g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})}{f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})}\left(\prod_{l=j+1}^{k} \frac{f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{a})}{f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{b})}\right)\right| \\
&= \left(\prod_{l=1}^{j-1} \frac{|g_{i_l,y}(S_{i_{l+1}\cdots i_k}\mathbf{a})|}{|f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{b})|}\right)\frac{|g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}\frac{|f_{i_{j+1}\cdots i_k,x}(\mathbf{a})|}{|f_{i_{j+1}\cdots i_k,x}(\mathbf{b})|} \\
&\le \eta^{j-1}\frac{1}{d}R,
\end{aligned}$$

using (3.4.3) and where R is as in (3.4.10). Hence by (3.4.7)

$$\left|\frac{g_{\mathbf{i},x}(\mathbf{a})}{f_{\mathbf{i},x}(\mathbf{b})}\right| = \left|\frac{\sum_{j=1}^{k} G_j(\mathbf{a})}{f_{\mathbf{i},x}(\mathbf{b})}\right| \le \sum_{j=1}^{k}\left|\frac{G_j(\mathbf{a})}{f_{\mathbf{i},x}(\mathbf{b})}\right| \le \sum_{j=1}^{k}\frac{R}{d}\eta^{j-1} < \frac{R}{d(1-\eta)},$$

giving (3.4.12) with $C = R/d(1-\eta)$. □

We now consider the singular values of the Jacobian matrices. For $\mathbf{a} = (a_1, a_2) \in [0,1]^2$ write $\alpha_1(D_{\mathbf{a}}S_{\mathbf{i}}) \ge \alpha_2(D_{\mathbf{a}}S_{\mathbf{i}})$ for the singular values of $D_{\mathbf{a}}S_{\mathbf{i}}$.

Lemma 3.4.13. *There exists a constant $M \ge 1$ such that for all $\mathbf{a} \in [0,1]^2$ and $\mathbf{i} \in \mathcal{I}^*$ the singular values of the Jacobian matrices $D_{\mathbf{a}}S_{\mathbf{i}}$ satisfy*

$$M^{-1} \le \frac{\alpha_1(D_{\mathbf{a}}S_{\mathbf{i}})}{|f_{\mathbf{i},x}(\mathbf{a})|}, \frac{\alpha_2(D_{\mathbf{a}}S_{\mathbf{i}})}{|g_{\mathbf{i},y}(\mathbf{a})|} \le M. \tag{3.4.14}$$

Proof. Let

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

be a matrix with $|c| \le |a|$ and $|b| \le C|a|$ for some constant $C > 0$. Calculating the larger singular value $\alpha_1(A)$ of A gives

$$\alpha_1(A)^2 = \tfrac{1}{2}\Big((a^2+b^2+c^2) + \big((a^2+b^2+c^2)^2 - 4a^2c^2\big)^{1/2}\Big).$$

Making obvious estimates,

$$\tfrac{1}{2}a^2 \ \le\ \alpha_1(A)^2 \ \le\ a^2+b^2+c^2 \ \le\ (2+C^2)a^2.$$

Applying this to the matrix

$$D_{\mathbf{a}}S_{\mathrm{i}} = \begin{pmatrix} f_{\mathrm{i},x}(\mathbf{a}) & 0 \\ g_{\mathrm{i},x}(\mathbf{a}) & g_{\mathrm{i},y}(\mathbf{a}) \end{pmatrix},$$

where $|g_{\mathrm{i},y}(\mathbf{a})| \le |f_{\mathrm{i},x}(\mathbf{a})|$ by (3.4.6) and $|g_{\mathrm{i},x}(\mathbf{a})| \le C|f_{\mathrm{i},x}(\mathbf{a})|$ by (3.4.12), gives the inequality (3.4.14) for the ratio of $\alpha_1\left(D_{\mathbf{a}}S_{\mathrm{i}}\right)$ and $|f_{\mathrm{i},x}(\mathbf{a})|$. Using that $\alpha_1(A)\alpha_2(A) = |\det A| = |a||c|$ for the matrix A immediately gives the inequality (3.4.14) for the ratio of $\alpha_2\left(D_{\mathbf{a}}S_{\mathrm{i}}\right)$ and $|g_{\mathrm{i},y}(\mathbf{a})|$. □

A immediate consequence of Lemmas 3.4.9 and 3.4.13 is that the singular values of the Jacobian matrices satisfy bounded distortion. More formally, this means that there exists a constant $A \ge 1$ such that for all $\mathbf{a}, \mathbf{b} \in [0,1]^2$ and $\mathbf{i} \in \mathcal{I}^*$,

$$A^{-1} \ \le\ \frac{\alpha_1\left(D_{\mathbf{a}}S_{\mathrm{i}}\right)}{\alpha_1\left(D_{\mathbf{b}}S_{\mathrm{i}}\right)},\ \frac{\alpha_2\left(D_{\mathbf{a}}S_{\mathrm{i}}\right)}{\alpha_2\left(D_{\mathbf{b}}S_{\mathrm{i}}\right)} \ \le\ A. \tag{3.4.15}$$

We define the projection map onto the x-axis $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi(x,y) = x$. It is immediate that the projection of the nonlinear measure μ onto the x-axis, $\pi(\mu) = \mu \circ \pi^{-1}$, is a self-conformal measure. It follows from a result of Peres and Solomyak [46, Theorem

1.1] that the L^q-spectra of this projected measure, which we denote by

$$\beta(q) := \tau_{\pi(\mu)}(q),$$

exists for $q \geq 0$. This holds even if there are complicated overlaps between the components of the projected measure, which is the typical situation for us. It is worth mentioning that Peres and Solomyak's theorem is only stated for self-conformal measures and does not address any other measures supported on self-conformal sets. As such this is the only obstacle preventing us from including Gibbs and quasi-Bernoulli measures in our analysis. Although we expect that Peres and Solomyak's result can be generalised to include these we do not pursue this here.

For $s \in \mathbb{R}$, $q \geq 0$ and $\mathbf{a} \in [0,1]^2$, we define the *q-modified singular value function*, $\psi_{\mathbf{a}}^{s,q} : \mathcal{I}^* \to (0,\infty)$ by

$$\psi_{\mathbf{a}}^{s,q}(\mathtt{i}) = p(\mathtt{i})^q \alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})^{\beta(q)} \alpha_2(D_{\mathbf{a}}S_{\mathtt{i}})^{s-\beta(q)}. \tag{3.4.16}$$

It follows from (3.4.15) that for all $\mathbf{a}, \mathbf{b} \in [0,1]^2$ and $\mathtt{i} \in \mathcal{I}^*$ that the q-modified singular value function $\psi_{\mathbf{a}}^{s,q}(\mathtt{i}) \asymp_{s,q} \psi_{\mathbf{b}}^{s,q}(\mathtt{i})$. Moreover, by Lemma 3.4.9,

$$\psi_{\mathbf{a}}^{s,q}(\mathtt{i}) \asymp_{s,q} p(\mathtt{i})^q |f_{\mathtt{i},x}(\mathbf{a})|^{\beta(q)} |g_{\mathtt{i},y}(\mathbf{a})|^{s-\beta(q)}. \tag{3.4.17}$$

For each $k \in \mathbb{N}$ define $\Psi_{\mathbf{a},k}^{s,q}$ by

$$\Psi_{\mathbf{a},k}^{s,q} = \sum_{\mathtt{i} \in \mathcal{I}^k} \psi_{\mathbf{a}}^{s,q}(\mathtt{i}).$$

The quantities $\psi_{\mathbf{a}}^{s,q}(\mathtt{i})$ and $\Psi_{\mathbf{a},k}^{s,q}$ satisfy some useful multiplicative properties, similar to those from [25, Lemma 2.2].

Lemma 3.4.18. *Let $s \in \mathbb{R}$, $q \geq 0$ and $\mathbf{a} \in [0,1]^2$.*

(a) *If $\mathtt{i}, \mathtt{j} \in \mathcal{I}^*$ then*

$$\psi_{\mathbf{a}}^{s,q}(\mathtt{ij}) \asymp_{s,q} \psi_{\mathbf{a}}^{s,q}(\mathtt{i})\psi_{\mathbf{a}}^{s,q}(\mathtt{j}). \tag{3.4.19}$$

(b) *If $k, l \in \mathbb{N}$ then*

$$\Psi_{\mathbf{a},k+l}^{s,q} \asymp_{s,q} \Psi_{\mathbf{a},k}^{s,q}\Psi_{\mathbf{a},l}^{s,q}. \tag{3.4.20}$$

Proof. By the chain rule applied to $f_{\mathtt{ij},x}$ and using (3.4.10),

$$|f_{\mathtt{ij},x}(\mathbf{a})| = |f_{\mathtt{i},x}(S_{\mathtt{i}}\mathbf{a})||f_{\mathtt{j},x}(\mathbf{a})| \asymp |f_{\mathtt{i},x}(\mathbf{a})||f_{\mathtt{j},x}(\mathbf{a})|,$$

and similarly

$$|g_{\mathtt{ij},y}(\mathbf{a})| \asymp |g_{\mathtt{i},y}(\mathbf{a})||g_{\mathtt{j},y}(\mathbf{a})|.$$

Using the form (3.4.17)

$$\begin{aligned}
\psi_{\mathbf{a}}^{s,q}(\mathtt{ij}) &\asymp_{s,q} p(\mathtt{ij})^q |f_{\mathtt{ij},x}(\mathbf{a})|^{\beta(q)}|g_{\mathtt{ij},y}(\mathbf{a})|^{s-\beta(q)} \\
&\asymp_{s,q} p(\mathtt{i})^q p(\mathtt{j})^q |f_{\mathtt{i},x}(\mathbf{a})|^{\beta(q)}|f_{\mathtt{j},x}(\mathbf{a})|^{\beta(q)}|g_{\mathtt{i},y}(\mathbf{a})|^{s-\beta(q)}|g_{\mathtt{j},y}(\mathbf{a})|^{s-\beta(q)} \\
&\asymp_{s,q} \psi_{\mathbf{a}}^{s,q}(\mathtt{i})\psi_{\mathbf{a}}^{s,q}(\mathtt{j}),
\end{aligned}$$

giving (3.4.19)

For part (b), if $k, l \in \mathbb{N}$ then

$$\Psi_{\mathbf{a},k+l}^{s,q} = \sum_{\mathtt{i}\in\mathcal{I}^{k+l}} \psi_{\mathbf{a}}^{s,q}(\mathtt{i}) = \sum_{\mathtt{i}\in\mathcal{I}^{k}}\sum_{\mathtt{j}\in\mathcal{I}^{l}} \psi_{\mathbf{a}}^{s,q}(\mathtt{ij})$$

and

$$\Psi_{\mathbf{a},k}^{s,q}\Psi_{\mathbf{a},l}^{s,q} = \Big(\sum_{\mathtt{i}\in\mathcal{I}^{k}} \psi_{\mathbf{a}}^{s,q}(\mathtt{i})\Big)\Big(\sum_{\mathtt{j}\in\mathcal{I}^{l}} \psi_{\mathbf{a}}^{s,q}(\mathtt{j})\Big) = \sum_{\mathtt{i}\in\mathcal{I}^{k}}\sum_{\mathtt{j}\in\mathcal{I}^{l}} \psi_{\mathbf{a}}^{s,q}(\mathtt{i})\psi_{\mathbf{a}}^{s,q}(\mathtt{j}).$$

Applying part (a) to the double sums completes the proof. $\square$

We call a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n > 0$ (such as those in Lemma 3.4.18) for which there exists an absolute constants $0 < K_1 \le K_2$ such that

$$K_1 a_n a_m \le a_{n+m} \le K_2 a_n a_m \tag{3.4.21}$$

for all $n, m \in \mathbb{N}$ *almost-multiplicative.* For such sequences the limit $\lim_{n\to\infty} a_n^{1/n}$ exists, see for example [13, Corollary 1.2] (it is worth noting that this limit exists even if the only inequality in (3.4.21) which holds is $a_{n+m} \le K_2 a_n a_m$).

It follows from Lemma 3.4.18 that for each $\mathbf{a} \in [0,1]^2$ we may define a function $P_{\mathbf{a}} : \mathbb{R} \times [0,\infty) \to [0,\infty)$ by

$$P_{\mathbf{a}}(s,q) = \lim_{k\to\infty} (\Psi^{s,q}_{\mathbf{a},k})^{1/k}.$$

The value of $P_{\mathbf{a}}(s,q)$ is unchanged if we replace the right-hand side of (3.4.16) by the right-hand side of (3.4.17) in the definition of $\psi^{s,q}_{\mathbf{a}}(\mathbf{i})$ and thus of $\Psi^{s,q}_{\mathbf{a},k}$. Moreover, as $\psi^{s,q}_{\mathbf{a}}(\mathbf{i}) \asymp_{s,q} \psi^{s,q}_{\mathbf{b}}(\mathbf{i})$ and thus $\Psi^{s,q}_{\mathbf{a},k} \asymp_{s,q} \Psi^{s,q}_{\mathbf{b},k}$ for all $\mathbf{a}, \mathbf{b} \in [0,1]^2$ it is easy to see that $P_{\mathbf{a}}$ is independent of the choice of $\mathbf{a}$. Thus we shall just write P instead of $P_{\mathbf{a}}$. For a fixed $q \ge 0$ we think of the function $s \mapsto \log P(s,q)$ as the *topological pressure* of the system.

We also write the following

$$\begin{aligned}
\alpha_{\min} &= \inf\{\alpha_2(D_{\mathbf{a}}S_i) : \mathbf{a} \in [0,1]^2,\ i \in \mathcal{I}\},\\
\alpha_{\max} &= \sup\{\alpha_1(D_{\mathbf{a}}S_i) : \mathbf{a} \in [0,1]^2,\ i \in \mathcal{I}\},\\
p_{\min} &= \min\{p_i : i \in \mathcal{I}\},\\
p_{\max} &= \max\{p_i : i \in \mathcal{I}\}
\end{aligned}$$

and note that $0 < \alpha_{\min}, \alpha_{\max}, p_{\min}, p_{\max} < 1$.

Recall that the L^q-spectrum of a given measure is Lipschitz continuous (as it is concave and decreasing) on $[\lambda, \infty)$ for all $\lambda > 0$. Let L_λ denote the Lipschitz constant of β on $[\lambda, \infty)$. We can now state some basic properties of P.

Lemma 3.4.22. *(1) For $s, r \in \mathbb{R}$ and $\lambda > 0$ define*

$$U(s, r, \lambda) = \min \left\{ \alpha_{\min}^s p_{\min}^r, \alpha_{\min}^s p_{\max}^r, \alpha_{\max}^s p_{\min}^r, \alpha_{\max}^s p_{\max}^r \right\} (\alpha_{\max}/\alpha_{\min})^{\min\{-L_\lambda r, 0\}}$$

and

$$V(s, r, \lambda) = \max \left\{ \alpha_{\min}^s p_{\min}^r, \alpha_{\min}^s p_{\max}^r, \alpha_{\max}^s p_{\min}^r, \alpha_{\max}^s p_{\max}^r \right\} (\alpha_{\max}/\alpha_{\min})^{\max\{-L_\lambda r, 0\}}.$$

Then for all $s, t \in \mathbb{R}, \lambda > 0, q \geq \lambda$ and $r \geq \lambda - q$

$$U(s, r, \lambda) P(t, q) \leq P(s + t, q + r) \leq V(s, r, \lambda) P(t, q),$$

and for all $s, t \in \mathbb{R}$

$$\min \left\{ \alpha_{\min}^s, \alpha_{\max}^s \right\} P(t, 0) \leq P(s + t, 0) \leq \max \left\{ \alpha_{\min}^s, \alpha_{\max}^s \right\} P(t, 0).$$

Also for all $s \in \mathbb{R}$ and $q \geq 0$,

$$P(s, q) \leq p_{\max}^q P(s, 0).$$

(2) P is continuous on $\mathbb{R} \times (0, \infty)$ and on $\mathbb{R} \times \{0\}$;

(3) P is strictly decreasing in $s \in \mathbb{R}$ and $q \in (0, \infty)$;

(4) For each $q \geq 0$ there exists a unique $s \geq 0$ such that $P(s, q) = 1$.

Proof. This is essentially the same as the proof of the analogous result of Fraser [25, Lemma 2.3] and as such is omitted. □

It follows from Lemma 3.4.22 that we may define a function $\gamma : [0, \infty) \to \mathbb{R}$ by $P(\gamma(q), q) = 1$. This function satisfies the following useful properties.

Lemma 3.4.23.

(1) γ *is strictly decreasing on* $[0, \infty)$;

(2) γ *is continuous on* $(0, \infty)$;

(3) $\gamma(1) = 0$ *and* $\lim_{q\to\infty} \gamma(q) = -\infty$;

(4) γ *is convex on* $(0, \infty)$.

Proof. This follows by the same reasoning as in the proof of [25, Lemma 2.5]. □

We can now state our main theorem which relates γ to the L^q-spectrum $\tau_\mu(q)$ of μ.

Theorem 3.4.24. *Let* μ *be a nonlinear measure which satisfies the domination condition (Definition 3.4.1). Then*

(1) For $q \in [0, 1]$

$$\overline{\tau}_\mu(q) \leq \gamma(q);$$

(2) For $q \geq 1$

$$\underline{\tau}_\mu(q) \geq \gamma(q);$$

(3) If μ *also satisfies the ROSC then for all* $q \geq 0$

$$\tau_\mu(q) = \gamma(q).$$

We shall prove this theorem in Section 3.5.

As a corollary we are able to calculate the box dimension (Definition 1.3.1) of the support of these measures.

Corollary 3.4.25. *Let F be a nonlinear attractor which satisfies the domination condition (Definition 3.4.1). Then (1)*

$$\overline{\dim}_B F \le \gamma(0);$$

(2) If F also satisfies the ROSC then

$$\dim_B F = \gamma(0).$$

Proof. We know that the upper and lower box dimension of the support of a measure is given by the upper and lower L^q-spectrum at 0. The result is then immediate from Theorem 3.4.24. □

Note that $\gamma(0)$ depends on $\beta(0) = \dim_B \pi F$, the box dimension of the projection of F onto the x-axis. Also, by standard results, e.g. [16, Corollary 3.10], the *packing* dimension of a nonlinear attractor coincides with the upper box dimension and so Corollary 3.4.25 also yields the packing dimension.

3.5 Calculating the L^q-spectrum

We begin this section by introducing some notation. For $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}^*$ let $\hat{\mathbf{i}} \in \mathcal{I}^* \cup \{\omega\}$ be given by

$$\hat{\mathbf{i}} = (i_1, i_2, \ldots, i_{k-1}),$$

where ω is the empty word. For $\delta \in (0,1]$ and $\mathbf{a} \in [0,1]^2$ we define the δ-*stopping* by

$$\mathcal{I}_{\mathbf{a},\delta} = \{\mathtt{i} \in \mathcal{I}^* : \alpha_2(D_{\mathbf{a}}S_{\mathtt{i}}) < \delta \leq \alpha_2(D_{\mathbf{a}}S_{\hat{\mathtt{i}}})\},$$

where S_ω is the identity map. If $\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}$ then

$$\alpha_{\min}\delta \leq \alpha_2(D_{\mathbf{a}}S_{\mathtt{i}}) < \delta. \tag{3.5.1}$$

For $\mathtt{i} \in \mathcal{I}^*$ let $\mu_{\mathtt{i}} = p(\mathtt{i})\mu \circ S_{\mathtt{i}}^{-1}$ and $F_{\mathtt{i}} = S_{\mathtt{i}}(F) = \mathrm{supp}\mu_{\mathtt{i}}$. Note that for all $\mathbf{a} \in [0,1]^2$ and $\delta \in (0,1]$,

$$\mu = \sum_{\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}} \mu_{\mathtt{i}}.$$

Lemma 3.5.2. *Let $\mathbf{a} \in [0,1]^2$, $t \in \mathbb{R}$ and $q \geq 0$.*

(1) If $t > \gamma(q)$ then

$$\sum_{\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}} \psi_{\mathbf{a}}^{t,q}(\mathtt{i}) \lesssim_{t,q} 1$$

for all $\delta \in (0,1]$.

(2) If $t < \gamma(q)$ then

$$\sum_{\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}} \psi_{\mathbf{a}}^{t,q}(\mathtt{i}) \gtrsim_{t,q} 1$$

for all $\delta \in (0,1]$.

Proof. The proof follows that of [25, Lemma 7.1] which only depends on the multiplicative properties of Ψ (which we have established here) so is omitted. $\square$

Our next lemma allows us to control the length of the side of $S_{\mathtt{i}}([0,1]^2)$ in terms of the length of its base.

Lemma 3.5.3. *There exists $L > 0$ such that for all $\mathtt{i} \in \mathcal{I}^*$ and all $0 \leq a < b \leq 1$*

$$\frac{|g_{\mathtt{i}}(b,0) - g_{\mathtt{i}}(a,0)|}{|f_{\mathtt{i}}(b) - f_{\mathtt{i}}(a)|} \leq L, \qquad \frac{|g_{\mathtt{i}}(b,1) - g_{\mathtt{i}}(a,1)|}{|f_{\mathtt{i}}(b) - f_{\mathtt{i}}(a)|} \leq L,$$

noting that $f_{\mathtt{i}}$ is a function of a single argument.

Proof. By the mean value theorem there exist $c_1, c_2 \in (a,b)$ such that

$$\frac{|g_{\mathtt{i}}(b,0) - g_{\mathtt{i}}(a,0)|}{|f_{\mathtt{i}}(b) - f_{\mathtt{i}}(a)|} = \frac{|g_{\mathtt{i},x}(c_1,0)||b-a|}{|f_{\mathtt{i},x}(c_2)||b-a|} = \frac{|g_{\mathtt{i},x}(c_1,0)|}{|f_{\mathtt{i},x}(c_2)|} \lesssim 1$$

by Lemma 3.4.11. The same reasoning holds on replacing 0 with 1. Taking L to be the maximum of the two implied constants completes the result. $\square$

The standard inequalities, that if $k \in \mathbb{N}, a_1, \ldots, a_k \geq 0$ and $q \geq 0$ then

$$\left(\sum_{i=1}^{k} a_i\right)^q \asymp_{k,q} \sum_{i=1}^{k} a_i^q, \tag{3.5.4}$$

will be helpful when manipulating moment sums.

Recall from (1.6.2) that $\mathcal{D}_\delta^q$ denotes the q-th power moment sum of a measure over the δ-mesh cubes $\mathcal{D}_\delta$. Our next result compares the moment sums of $\mu_{\mathtt{i}} = p(\mathtt{i})\mu \circ S_{\mathtt{i}}^{-1}$ on $S_{\mathtt{i}}(F)$ with moment sums of the projection of μ onto the horizontal axis. This is analogous to [25, Lemma 7.2] but in the nonlinear case more care is needed.

Lemma 3.5.5. *For each $q \geq 0$ and $\mathbf{a} \in [0,1]^2$, there exist numbers $\widehat{A}, \widehat{B} > 0$ such that if we write*

$$\widehat{A}_{\mathtt{i},\delta} = \frac{\widehat{A}\delta}{\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})} \qquad \textit{and} \qquad \widehat{B}_{\mathtt{i},\delta} = \frac{\widehat{B}\delta}{\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})} \tag{3.5.6}$$

for $\delta \in (0,1]$ *and* $\mathbf{i} \in \mathcal{I}_{\mathbf{a},\delta}$ *then*

$$\mathcal{D}^q_{\widehat{B}_{\mathrm{i},\delta}}(p(\mathbf{i})\pi\mu) \lesssim \mathcal{D}^q_\delta(\mu_{\mathrm{i}}) \lesssim \mathcal{D}^q_{\widehat{A}_{\mathrm{i},\delta}}(p(\mathbf{i})\pi\mu). \tag{3.5.7}$$

Proof. As $\mathbf{i} \in \mathcal{I}_{\mathbf{a},\delta}$ we have $\alpha_2(D_{\mathbf{a}}S_{\mathrm{i}}) < \delta$. We shall show that there are at most a constant number k squares of the δ-mesh that intersect $S_{\mathrm{i}}([0,1]^2) \supseteq \mathrm{supp}\mu_{\mathrm{i}}$ in any vertical column of mesh squares.

For this we estimate the height of the intersection of $S_{\mathrm{i}}([0,1]^2)$ with a given vertical strip of width δ. Note that for any such vertical strip there exists some $0 \leq a < b \leq 1$ such that

$$|f_{\mathrm{i}}(b) - f_{\mathrm{i}}(a)| = \delta, \tag{3.5.8}$$

apart from at most two vertical strips (at the left and right ends of $S_{\mathrm{i}}([0,1]^2)$) for which

$$|f_{\mathrm{i}}(b) - f_{\mathrm{i}}(a)| \leq \delta \tag{3.5.9}$$

with one of a or b equal to either 0 or 1 (this is displayed in Figure 3.2). Then, for $a', b' \in [a,b]$,

$$\begin{aligned} |g_{\mathrm{i}}(b',1) - g_{\mathrm{i}}(a',0)| &\leq |g_{\mathrm{i}}(b',1) - g_{\mathrm{i}}(a',1)| + |g_{\mathrm{i}}(a',1) - g_{\mathrm{i}}(a',0)| \qquad (3.5.10)\\ &\leq L|f_{\mathrm{i}}(b') - f_{\mathrm{i}}(a')| + |g_{\mathrm{i},y}(a',c)| \\ &\lesssim L\delta + \alpha_2(D_{(a',c)}S_{\mathrm{i}}) \\ &\lesssim \delta, \end{aligned}$$

where we have estimated the first term of (3.5.10) using Lemma 3.5.3 and (3.5.8)-(3.5.9) and the second term using the mean value theorem with $c \in (0,1)$ followed by (3.4.14), (3.4.15) and (3.5.1).

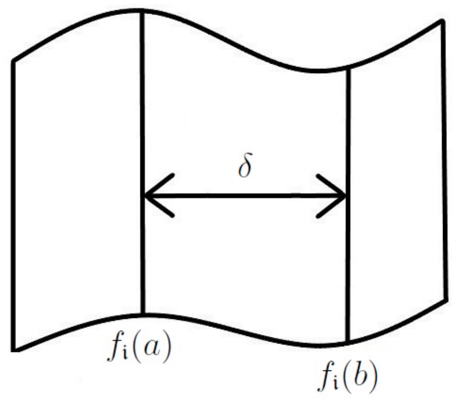

Figure 3.2: $S_{\mathtt{i}}([0,1]^2)$ together with two points $f_{\mathtt{i}}(a)$ and $f_{\mathtt{i}}(b)$ which together form a vertical strip of width δ.

We have shown that the height of the intersection of $S_{\mathtt{i}}([0,1]^2)$ with every vertical strip with base length δ is at most $k'\delta$, where k' is independent of $\mathtt{i}$. Thus at most $k = \lceil k' \rceil + 1$ squares in any column of the δ-mesh intersect $S_{\mathtt{i}}([0,1]^2)$ so using (3.5.4) $\mathcal{D}_\delta^q(\mu_{\mathtt{i}}) \asymp \mathcal{D}_\delta^q(\pi\mu_{\mathtt{i}})$ where $\pi\mu_{\mathtt{i}}$ is the projection of $\mu_{\mathtt{i}}$ onto the x-axis. In terms of the projection of pre-images of the intersection of δ-mesh cubes Q with $S_{\mathtt{i}}([0,1]^2)$,

$$\begin{aligned}
\mathcal{D}_\delta^q(\mu_{\mathtt{i}}) &= \sum_{Q \in \mathcal{D}_\delta} \mu_{\mathtt{i}} \left(Q \cap S_{\mathtt{i}}([0,1]^2) \right)^q \\
&= p(\mathtt{i})^q \sum_{Q \in \mathcal{D}_\delta} \mu \left(S_{\mathtt{i}}^{-1} \left(Q \cap S_{\mathtt{i}}([0,1]^2) \right) \right)^q \\
&\asymp p(\mathtt{i})^q \sum_{Q \in \mathcal{D}_\delta} \pi\mu \left(\pi S_{\mathtt{i}}^{-1} \left(Q \cap S_{\mathtt{i}}([0,1]^2) \right) \right)^q . \qquad (3.5.11)
\end{aligned}$$

If Q is a δ-mesh cube that intersects $S_{\mathtt{i}}([0,1]^2)$ other than one overlapping its left or right edge, then $\pi\left(Q \cap S_{\mathtt{i}}([0,1]^2)\right)$ is an interval in $\mathbb{R}$ of length δ. Writing $\hat{a}, \hat{b}$ for the endpoints of $\pi S_{\mathtt{i}}^{-1}\left(Q \cap S_{\mathtt{i}}([0,1]^2)\right)$ then using the mean value theorem and (3.4.14),

$$|\hat{a} - \hat{b}| \;=\; \frac{|f_{\mathtt{i}}(\hat{a}) - f_{\mathtt{i}}(\hat{b})|}{|f_{\mathtt{i},x}(\hat{c})|} \;=\; \frac{\delta}{|f_{\mathtt{i},x}(\hat{c})|} \;\asymp\; \frac{\delta}{\alpha_1(D_{\mathbf{a}} S_{\mathtt{i}})},$$

where $\hat{c} \in [0,1]$.

If Q is one of the two cubes at the left or right end of $S_{\mathtt{i}}([0,1]^2)$ then we simply "glue" $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ to the adjacent interval (which will be on the right or left respectively). This will create a new interval which also has length comparable to $\delta/\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})$.

Every projection of a pre-image $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ can be covered by an interval of length $\widehat{A}\delta/\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})$ and contains an interval of length $\widehat{B}\delta/\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})$, for some constants $\widehat{A} \geq \widehat{B} > 0$. Recall the definitions (3.5.6) of $\widehat{A}_{\mathtt{i},\delta}$ and $\widehat{B}_{\mathtt{i},\delta}$ and write $\mathcal{J}_\delta$ for the δ-mesh on $\mathbb{R}$ centred at the origin. From (3.5.11), noting that each $J \in \mathcal{J}_{\widehat{A}_{\mathtt{i},\delta}}$ can intersect $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ for at most $k\left\lceil \widehat{A}/\widehat{B} + 1\right\rceil$ many $Q \in \mathcal{D}_\delta$, and using (3.5.4),

$$\begin{aligned}
\mathcal{D}_\delta^q(\mu_{\mathtt{i}}) &\asymp p(\mathtt{i})^q \sum_{Q \in \mathcal{D}_\delta} \pi\mu\left(\pi S_{\mathtt{i}}^{-1}\left(Q \cap S_{\mathtt{i}}([0,1]^2)\right)\right)^q \\
&\lesssim p(\mathtt{i})^q \sum_{J \in \mathcal{J}_{\widehat{A}_{\mathtt{i},\delta}}} \pi\mu(J)^q \\
&= \sum_{J \in \mathcal{J}_{\widehat{A}_{\mathtt{i},\delta}}} (p(\mathtt{i})\pi\mu(J))^q \\
&= \mathcal{D}_{\widehat{A}_{\mathtt{i},\delta}}^q(p(\mathtt{i})\pi\mu).
\end{aligned}$$

Similarly, each $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ intersects at most $\left\lceil \widehat{A}/\widehat{B} + 1\right\rceil$ intervals $J \in \mathcal{J}_{\widehat{B}_{\mathtt{i},\delta}}$, and each interval $J \in \mathcal{J}_{\widehat{B}_{\mathtt{i},\delta}}$ intersects $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ for at most $2k$ sets $Q \in \mathcal{D}_\delta$, so

$$\begin{aligned}
\mathcal{D}_\delta^q(\mu_{\mathtt{i}}) &\asymp p(\mathtt{i})^q \sum_{Q \in \mathcal{D}_\delta} \pi\mu\left(\pi S_{\mathtt{i}}^{-1}\left(Q \cap S_{\mathtt{i}}([0,1]^2)\right)\right)^q \\
&\gtrsim p(\mathtt{i})^q \sum_{J \in \mathcal{J}_{\widehat{B}_{\mathtt{i},\delta}}} \pi\mu(J)^q
\end{aligned}$$

$$= \sum_{J \in \mathcal{J}_{\widehat{B}_{\mathtt{i},\delta}}} (p(\mathtt{i})\pi\mu(J))^q$$

$$= \mathcal{D}^q_{\widehat{B}_{\mathtt{i},\delta}}(p(\mathtt{i})\pi\mu),$$

giving the result. □

Notice that a simple consequence of the Definition 1.6.3 of the L^q-spectrum is that for all $\varepsilon > 0, q \geq 0, p > 0$ and $0 < \delta \leq 1$,

$$p^q \delta^{-\beta(q)+\varepsilon/2} \lesssim_{\varepsilon,q} \mathcal{D}^q_\delta(p\pi\mu) \lesssim_{\varepsilon,q} p^q \delta^{-\beta(q)-\varepsilon/2}. \tag{3.5.12}$$

We now turn to proving our main result, Theorem 3.4.24.

Proof of Theorem 3.4.24. The first two parts of this proof follow Fraser's proof of [25, Theorem 2.6] but we reproduce it here due to its centrality to our result.

Part (1). Let $q \in [0,1]$ and let $\delta \in (0,1]$ and $\mathbf{a} \in [0,1]^2$. It is sufficient to show that $\overline{\tau}_\mu(q) \leq \gamma(q)$. As $q \in [0,1]$,

$$\mathcal{D}^q_\delta(\mu) = \sum_{Q \in \mathcal{D}_\delta} \mu(Q)^q = \sum_{Q \in \mathcal{D}_\delta} \left(\sum_{\mathtt{i} \in \mathcal{I}_\delta} \mu_\mathtt{i}(Q) \right)^q$$

$$\leq \sum_{Q \in \mathcal{D}_\delta} \sum_{\mathtt{i} \in \mathcal{I}_\delta} \mu_\mathtt{i}(Q)^q = \sum_{\mathtt{i} \in \mathcal{I}_\delta} \sum_{Q \in \mathcal{D}_\delta} \mu_\mathtt{i}(Q)^q = \sum_{\mathtt{i} \in \mathcal{I}_\delta} \mathcal{D}^q_\delta(\mu_\mathtt{i}).$$

Thus for all $\varepsilon > 0$,

$$\begin{aligned} \delta^{\gamma(q)+\varepsilon}\mathcal{D}^q_\delta(\mu) &\leq \delta^{\gamma(q)+\varepsilon} \sum_{\mathtt{i} \in \mathcal{I}_\delta} \mathcal{D}^q_\delta(\mu_\mathtt{i}) \\ &\lesssim \delta^{\gamma(q)+\varepsilon} \sum_{\mathtt{i} \in \mathcal{I}_\delta} \mathcal{D}^q_{\widehat{A}_{\mathtt{i},\delta}}(p(\mathtt{i})\pi_\mathtt{i}\mu) && \text{by (3.5.7)} \end{aligned}$$

$$
\begin{aligned}
&\lesssim_{\varepsilon,q} \ \delta^{\gamma(q)+\varepsilon}\sum_{\mathtt{i}\in\mathcal{I}_\delta} p(\mathtt{i})^q\left(\frac{\widehat{A}\delta}{\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})}\right)^{-\beta(q)-\varepsilon/2} && \text{by (3.5.12)}\\
&\lesssim_{\varepsilon,q} \ \sum_{\mathtt{i}\in\mathcal{I}_\delta} p(\mathtt{i})^q\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})^{\beta(q)+\varepsilon/2}\alpha_2(D_{\mathbf{a}}S_{\mathtt{i}})^{\gamma(q)+\varepsilon-\beta(q)-\varepsilon/2} && \text{by (3.5.1)}\\
&= \ \sum_{\mathtt{i}\in\mathcal{I}_\delta} \psi_{\mathbf{a}}^{\gamma(q)+\varepsilon,q}(\mathtt{i}) && \text{by (3.4.16)}\\
&\lesssim_{\varepsilon,q} 1. && \text{by Lemma 3.5.2}
\end{aligned}
$$

So $\overline{\tau}_\mu(q) \leq \gamma(q)+\varepsilon$ by (1.6.4), giving (1) on letting $\varepsilon \to 0$.

Part (2). We suppose $q \geq 1$ and as before let $\delta \in (0,1]$ and $\mathbf{a} \in [0,1]^2$. It is sufficient to show that $\underline{\tau}_\mu(q) \geq \gamma(q)$. As $q \geq 1$,

$$
\mathcal{D}_\delta^q(\mu) = \sum_{Q\in\mathcal{D}_\delta} \mu(Q)^q = \sum_{Q\in\mathcal{D}_\delta}\left(\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mu_{\mathtt{i}}(Q)\right)^q
$$

$$
\geq \sum_{Q\in\mathcal{D}_\delta}\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mu_{\mathtt{i}}(Q)^q = \sum_{\mathtt{i}\in\mathcal{I}_\delta}\sum_{Q\in\mathcal{D}_\delta}\mu_{\mathtt{i}}(Q)^q = \sum_{\mathtt{i}\in\mathcal{I}_\delta}\mathcal{D}_\delta^q(\mu_{\mathtt{i}}).
$$

Thus for all $\varepsilon > 0$,

$$
\begin{aligned}
\delta^{\gamma(q)-\varepsilon}\mathcal{D}_\delta^q(\mu) \ &\geq \ \delta^{\gamma(q)-\varepsilon}\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mathcal{D}_\delta^q(\mu_{\mathtt{i}}) && \\
&\gtrsim \ \delta^{\gamma(q)-\varepsilon}\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mathcal{D}^q_{\widehat{B}_{\mathtt{i},\delta}}(p(\mathtt{i})\pi_{\mathtt{i}}\mu) && \text{by (3.5.7)}\\
&\gtrsim_{\varepsilon,q} \ \delta^{\gamma(q)-\varepsilon}\sum_{\mathtt{i}\in\mathcal{I}_\delta} p(\mathtt{i})^q\left(\frac{\widehat{B}\delta}{\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})}\right)^{-\beta(q)+\varepsilon/2} && \text{by (3.5.12)}\\
&\gtrsim_{\varepsilon,q} \ \sum_{\mathtt{i}\in\mathcal{I}_\delta} p(\mathtt{i})^q\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})^{\beta(q)-\varepsilon/2}\alpha_2(D_{\mathbf{a}}S_{\mathtt{i}})^{\gamma(q)-\varepsilon-\beta(q)+\varepsilon/2} && \text{by (3.5.1)}\\
&= \ \sum_{\mathtt{i}\in\mathcal{I}_\delta} \psi_{\mathbf{a}}^{\gamma(q)-\varepsilon,q}(\mathtt{i}) && \text{by (3.4.16)}\\
&\gtrsim_{\varepsilon,q} 1. && \text{by Lemma 3.5.2}
\end{aligned}
$$

So $\underline{\tau}_\mu(q) \geq \gamma(q) - \varepsilon$ giving (2) on letting $\varepsilon \to 0$.

Part (3). We now assume μ satisfies the ROSC. Due to Parts (1) and (2) we only need to provide an upper bound when $q > 1$ and a lower bound when $q < 1$.

We begin by looking at the case when $q > 1$. For (1) we obtained an upper bound when $q \in [0, 1]$, but the only place in the proof where we used the assumption $q \leq 1$ was

$$\mathcal{D}_\delta^q(\mu) \leq \sum_{\mathtt{i} \in \mathcal{I}_\delta} \mathcal{D}_\delta^q(\mu_{\mathtt{i}}).$$

Thus for $q > 1$ we shall use the ROSC to show that

$$\mathcal{D}_\delta^q(\mu) \lesssim \sum_{\mathtt{i} \in \mathcal{I}_\delta} \mathcal{D}_\delta^q(\mu_{\mathtt{i}}).$$

It follows from Hölder's inequality that for $Q \in \mathcal{D}_\delta$

$$\Big(\sum_{\mathtt{i} \in \mathcal{I}_\delta} \mu_{\mathtt{i}}(Q) \Big)^q \leq \; k^{q-1} \sum_{\mathtt{i} \in \mathcal{I}_\delta} \mu_{\mathtt{i}}(Q)^q,$$

where

$$k := |\{ \mathtt{i} \in \mathcal{I}_\delta : \mu_{\mathtt{i}}(Q) > 0 \}|. \tag{3.5.13}$$

To complete the proof we need to bound k uniformly for all δ and $Q \in \mathcal{D}_\delta$. Fix $\delta \in (0, 1]$ and $Q \in \mathcal{D}_\delta$ such that $\mu(Q) > 0$. For convenience if $A > 0$ then we write AQ to denote the cube with the same centre as Q but with sidelength $A\delta$.

Let $\mathtt{i} \in \mathcal{I}_\delta$ be such that $S_{\mathtt{i}}((0, 1)^2) \cap Q$ is non-empty (such an $\mathtt{i}$ must exist as by assumption $\mu(Q) > 0$). Let $\mathbf{a} \in S_{\mathtt{i}}((0, 1)^2) \cap Q$ and consider the vertical "slice" of $S_{\mathtt{i}}((0, 1)^2)$ that contains $\mathbf{a}$. By (3.5.1) and Lemma 3.4.13, $g_{\mathtt{i},y}(\mathbf{a}) \asymp \alpha_2(D_{\mathbf{a}} S_{\mathtt{i}}) \asymp \delta$. Together with the mean value theorem this implies that the height of this vertical slice is comparable to δ, say it is bounded above by $M\delta$ for some $M > 1$ which is independent of δ.

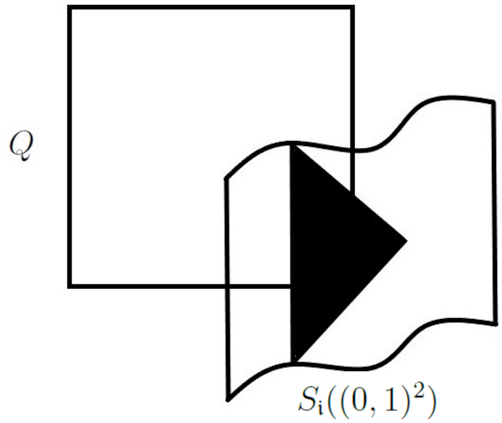

Figure 3.3: $S_{\mathtt{i}}((0,1)^2)$ and Q, together with the triangle $\Delta_{\mathtt{i}}$ contained in $S_{\mathtt{i}}((0,1)^2)$.

Lemma 3.5.3 implies that if we draw a line of slope L (where we can assume $L > 1$) from the base of the vertical slice in both directions and a line of slope $-L$ from the top of the vertical slice in both directions then of the two isosceles triangles formed by these lines and the vertical slice at least one must lie within $S_{\mathtt{i}}((0,1)^2)$. As the length of the vertical slice is comparable to δ the area of this triangle is comparable to δ^2. We write $\Delta_{\mathtt{i}}$ for the triangle which is contained in $S_{\mathtt{i}}((0,1)^2)$, see Figure 3.3.

Each triangle $\Delta_{\mathtt{i}}$ (associated with $\mathtt{i} \in \mathcal{I}_\delta$ such that $S_{\mathtt{i}}((0,1)^2) \cap Q \neq \emptyset$) is contained in the square which has the same centre as Q and sidelength $3M\delta$, i.e. the square $3MQ$. Let $\mathcal{L}$ denote two-dimensional Lebesgue measure.

As the area of each $\Delta_{\mathtt{i}}$ is comparable to δ^2 and the ROSC guarantees that the interiors of the $\Delta_{\mathtt{i}}$ are pairwise disjoint, it follows from (3.5.13) that

$$k\delta^2 \lesssim \sum_{\substack{\mathtt{i} \in \mathcal{I}_\delta: \\ S_{\mathtt{i}}((0,1)^2) \cap Q \neq \emptyset}} \mathcal{L}(\Delta_{\mathtt{i}}) \leq (3M\delta)^2 = 9M^2\delta^2.$$

Hence $k \lesssim 1$ completing the proof of the upper bound for $q > 1$.

When $0 \leq q < 1$ a similar approach to the $q > 1$ case above establishes that

$$\mathcal{D}_\delta^q(\mu) \gtrsim \sum_{\mathrm{i} \in \mathcal{I}_\delta} \mathcal{D}_\delta^q(\mu_{\mathrm{i}}).$$

We omit the proof which is very similar. $\square$

Chapter 4

Ledrappier-Young formulae for measures on nonlinear attractors

4.1 Background

In the dimension theory of measures, it is a problem of central interest to establish the exact dimensionality (Definition 1.5.2) of ergodic invariant measures supported on attractors of IFSs, and to provide a formula for the exact dimension. In many settings, the exact dimension has been shown to satisfy a formula in terms of Lyapunov exponents, various notions of entropy and the dimensions of projected measures.

In particular, if the maps S_i in the IFS are conformal and the IFS satisfies an additional separation condition, it is a classical result that any ergodic invariant measure μ supported on the attractor F is exact dimensional and its exact dimension is given by its measure-theoretic entropy over the Lyapunov exponent (see e.g. [6]). In the substantially more difficult case where no separation condition is assumed, Feng and Hu [21] generalised this by showing that any ergodic invariant measure μ supported on the attractor F is exact dimensional and its exact dimension is given by the projection entropy over the Lyapunov exponent. In this sense, exact dimensionality is understood in the conformal setting.

On the other hand, the question of whether every ergodic invariant measure supported on the attractor of a non-conformal IFS is exact dimensional is still very much open, and this question has recently received a lot of attention in the particular case where the maps S_i are all affine. Feng [20] has very recently shown that all ergodic invariant measures supported on the attractors of IFSs composed of affine maps are exact dimensional and satisfy a formula in terms of the Lyapunov exponents and conditional entropies. This answered a folklore open question in the fractal community and unified previous partial results obtained in [3, 26]. In the non-conformal setting, this formula for the exact dimension of μ is often called a "Ledrappier-Young formula", following the work of Ledrappier and Young on the dimension of invariant measures for C^2 diffeomorphisms on compact manifolds [37, 38].

While Feng's result settles the case of ergodic measures supported on self-affine sets, the more general case of ergodic measures supported on attractors of more general (i.e. nonlinear) non-conformal IFSs is still open. In fact, the only result in this direction that we are aware of is [20, Theorem 2.11], where Feng and Hu prove exact dimensionality of ergodic invariant measures supported on the attractors of IFSs which can be expressed as the direct product of IFSs composed of C^1 maps on $\mathbb{R}$. The fact that there is limited literature concerning the exact-dimensionality of measures supported on general non-conformal attractors reflects the wider challenge of understanding the dimension theory of nonlinear non-conformal attractors, although this appears to be an area of growing interest [9, 17, 22].

In this chapter we study (pushforward) quasi-Bernoulli measures supported on the attractors of nonlinear, non-conformal IFSs which were introduced in Chapter 3, and we show that these are exact dimensional and satisfy a Ledrappier-Young formula. To do this we adapt an approach introduced in [26] to the nonlinear setting.

4.2 Our setting and statement of results

We will study the family of IFSs $\{S_i\}_{i\in\mathcal{I}}$ with corresponding attractors F which were introduced in Chapter 3, namely those defined in Definition 3.2.1 which also satisfy the domination condition (Definition 3.4.1). The only difference will be the separation condition - in Chapter 3 we assumed the rectangular open set condition or ROSC (Definition 2.3.3) whereas in this chapter we assume the rectangular strong separation condition (RSSC).

Definition 4.2.1 (Rectangular strong separation condition)**.** We say that an IFS $\{S_i\}_{i\in\mathcal{I}}$ with attractor F satisfies the *rectangular strong separation condition* (RSSC) if the sets $\{S_i([0,1]^2)\}_{i\in\mathcal{I}}$ are pairwise disjoint.

Recall from Chapter 1 that we write $\Pi : \Sigma \to [0,1]^n$ for the natural projection map. We will study the pushforward measure $\mu = m \circ \Pi^{-1}$ for a quasi-Bernoulli, shift-invariant, ergodic measure m (Definition 1.7.4) noting that μ is supported on F.

A Ledrappier-Young formula is defined in terms of entropy, Lyapunov exponents and dimensions of projected measures. We shall use the form of entropy in Theorem 1.7.6, but the definition for Lyapunov exponents we saw in Chapter 1 (Theorem 1.7.9) was only in the self-affine setting.

Lyapunov exponents contain information about how much our system contracts in each direction and in the nonlinear setting they are defined in terms of the Jacobian matrices

$D_{\mathbf{a}}S_{\mathrm{i}|n}$, where $\mathbf{a} \in [0,1]^2$, $\mathbf{i} \in \Sigma$ and $n \in \mathbb{N}$, see (3.4.4). To guarantee their existence we again require Kingman's subadditive ergodic theorem (Theorem 1.7.8).

Definition 4.2.2 (Lyapunov exponents (nonlinear setting))**.** Let m be quasi-Bernoulli, shift-invariant and ergodic and let $\mu = m \circ \Pi^{-1}$. Then there exist constants $0 < \chi_1(\mu) \leq \chi_2(\mu)$ such that for m almost all $\mathbf{i} \in \Sigma$,

$$\chi_1(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log \alpha_1 \left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n} \right) \tag{4.2.3}$$

and

$$\chi_2(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log \alpha_2 \left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n} \right). \tag{4.2.4}$$

We call $\chi_1(\mu), \chi_2(\mu)$ the Lyapunov exponents of the system with respect to μ.

The final important ingredient in a Ledrappier-Young formula is the dimension of an appropriate projected measure. Let $\pi : [0,1]^2 \to [0,1]$ denote projection to the x-coordinate. Let $\pi(\mu) = \mu \circ \pi^{-1}$ denote the projected measure which is supported on πF, which is the attractor of the (possibly overlapping) conformal IFS $\{f_i\}_{i \in \mathcal{I}}$ on $[0,1]$. By [21, Theorem 2.8], $\pi(\mu)$ is exact dimensional. We denote its exact dimension by $\dim \pi(\mu)$.

We are now ready to state the main result of this chapter.

Theorem 4.2.5. *Let μ be a pushforward quasi-Bernoulli measure supported on F, where F is a nonlinear attractor (Definition 3.2.1) which satisfies the domination condition (Definition 3.4.1) and the RSSC (Definition 4.2.1). Then μ is exact dimensional and moreover its exact dimension* $\dim \mu$ *satisfies the following Ledrappier-Young for-*

mula

$$\dim \mu = \frac{h(\mu)}{\chi_2(\mu)} + \frac{\chi_2(\mu) - \chi_1(\mu)}{\chi_2(\mu)} \dim \pi(\mu).$$

4.3 Proofs

There are several properties of our IFS established in Chapter 3 that will be particularly useful in calculating Ledrappier-Young formulae. Results that we will use repeatedly throughout this section include Lemmas 3.4.9, 3.4.11 and 3.4.13, as well as (3.4.15).

These results immediately yield some useful information. Lemma 3.4.13, together with the domination condition, implies that the two Lyapunov exponents are distinct, $\chi_1(\mu) < \chi_2(\mu)$.

An easy but useful consequence of (3.4.15) this is that the Lyapunov exponents defined in Definition 4.2.2 may be expressed as

$$\chi_1(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log \alpha_1 \left(D_{a_n} S_{\mathfrak{i}|n} \right) \quad \text{and} \quad \chi_2(\mu) = -\lim_{n\to\infty} \frac{1}{n} \log \alpha_2 \left(D_{a_n} S_{\mathfrak{i}|n} \right) \qquad (4.3.1)$$

for any sequence $(a_n)_{n\in\mathbb{N}}$ in $[0,1]^2$, on the same set of $\mathfrak{i} \in \Sigma$ of full m-measure that was used in Definition 4.2.2.

We now turn to proving some key technical results. The following lemma allows us to estimate the μ-measure of a small "approximate square" in $[0,1]^2$ by the product of the m-measure of an appropriate cylinder and the $\pi(\mu)$-measure of the π-projection of the "blow up" of the "approximate square". It is worth noting that this lemma is the only place where the assumption that m is quasi-Bernoulli (Definition 1.7.4) is used.

For $r > 0$, $n \in \mathbb{N}$, $\mathbf{a} = (a_1, a_2) \in [0,1]^2$ and $\mathfrak{i} \in \Sigma$ such that $\Pi(\mathfrak{i}) = \mathbf{a}$ we write $B_n(\mathbf{a}, r)$ to denote the strip of points $\mathbf{b} = (b_1, b_2) \in S_{\mathfrak{i}|n}([0,1]^2)$ whose x co-ordinate satisfies

$|b_1 - a_1| \leq r/2$. By the RSSC, Π is an injective map and therefore $B_n(\mathbf{a}, r)$ is well defined. For $x \in \mathbb{R}$ and $r > 0$ we write $Q_1(x,r) = [x - \frac{r}{2}, x + \frac{r}{2}]$.

Lemma 4.3.2. *Let* $r > 0$, $n \in \mathbb{N}$, $\mathbf{a} = (a_1, a_2) \in F$ *and* $\mathtt{i} \in \Sigma$ *such that* $\Pi(\mathtt{i}) = \mathbf{a}$. *Then*

$$\mu(B_n(\mathbf{a}, r)) \\ \leq Lm([\mathtt{i}|n])\pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^n\mathtt{i})), \frac{Mr}{\min_{\mathbf{b}\in[0,1]^2} \alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|n}\right)}\right)\right) \tag{4.3.3}$$

and

$$\mu(B_n(\mathbf{a}, r)) \\ \geq L^{-1}m([\mathtt{i}|n])\pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^n\mathtt{i})), \frac{M^{-1}r}{\max_{\mathbf{b}\in[0,1]^2} \alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|n}\right)}\right)\right), \tag{4.3.4}$$

where L *is the constant from the quasi-Bernoulli property* (1.7.5) *and where* M *is as defined in Lemma 3.4.13.*

Proof. Let

$$\mathcal{J} = \mathcal{J}(\mathtt{i}, n, r) = \left\{\mathtt{j} \in \mathcal{I}^* : S_{\mathtt{i}|n\mathtt{j}}([0,1]^2) \subseteq B_n(\mathbf{a}, r) \text{ and } S_{\mathtt{i}|n\mathtt{j}^\dagger}([0,1]^2) \nsubseteq B_n(\mathbf{a}, r)\right\},$$

writing $\mathtt{j}^\dagger$ to denote $\mathtt{j}$ with the last symbol removed. It follows by our separation assumption that the sets $\{S_{\mathtt{i}|n\mathtt{j}}([0,1]^2)\}_{\mathtt{j}\in\mathcal{J}}$ are pairwise disjoint and exhaust $B_n(\mathbf{a}, r)$ in measure, therefore

$$\mu(B_n(\mathbf{a}, r)) = \sum_{\mathtt{j}\in\mathcal{J}} m([\mathtt{i}|n\mathtt{j}]).$$

Furthermore as m is quasi-Bernoulli (Definition 1.7.4), we get

$$L^{-1}m([\mathbf{i}|n])\sum_{\mathtt{j}\in\mathcal{J}} m([\mathtt{j}]) \leq \mu(B_n(\mathbf{a},r)) \leq Lm([\mathbf{i}|n])\sum_{\mathtt{j}\in\mathcal{J}} m([\mathtt{j}]). \tag{4.3.5}$$

Note that the sets $\{S_{\mathtt{j}}([0,1]^2)\}_{\mathtt{j}\in\mathcal{J}}$ are disjoint and exhaust $S_{\mathtt{i}|n}^{-1}(B_n(\mathbf{a},r))$ in measure. Moreover, since $S_{\mathtt{i}|n}^{-1}B_n(\mathbf{a},r)$ necessarily has height 1

$$\sum_{\mathtt{j}\in\mathcal{J}} m([\mathtt{j}]) = \mu(S_{\mathtt{i}|n}^{-1}B_n(\mathbf{a},r)) = \pi(\mu)(\pi S_{\mathtt{i}|n}^{-1}(B_n(\mathbf{a},r))).$$

Observe that $\pi S_{\mathtt{i}|n}^{-1}(\mathbf{a}) = \pi(\Pi(\sigma^n(\mathbf{i})))$. Writing $\hat{a}$ and $\hat{b}$ for the left and right endpoints of $\pi S_{\mathtt{i}|n}^{-1}(B_n(\mathbf{a},r))$, it follows from the mean value theorem that

$$|\pi(\Pi(\sigma^n\mathbf{i})) - \hat{a}| = \frac{|a_1 - f_{\mathtt{i}|n}(\hat{a})|}{|f_{\mathtt{i}|n,x}(\hat{c}_1)|} = \frac{r}{2|f_{\mathtt{i}|n,x}(\hat{c}_1)|}$$

for some $\hat{c}_1 \in [0,1]$ and

$$|\pi(\Pi(\sigma^n\mathbf{i})) - \hat{b}| = \frac{|a_1 - f_{\mathtt{i}|n}(\hat{b})|}{|f_{\mathtt{i}|n,x}(\hat{c}_2)|} = \frac{r}{2|f_{\mathtt{i}|n,x}(\hat{c}_2)|}$$

for some $\hat{c}_2 \in [0,1]$. By Lemma 3.4.13

$$\begin{aligned} \max\left\{\frac{r}{2|f_{\mathtt{i}|n,x}(\hat{c}_1)|}, \frac{r}{2|f_{\mathtt{i}|n,x}(\hat{c}_2)|}\right\} &\leq \frac{r}{2\min_{\mathbf{b}\in[0,1]^2}|f_{\mathtt{i}|n,x}(\mathbf{b})|} \\ &\leq \frac{Mr}{2\min_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|n}\right)} \end{aligned}$$

and

$$\min\left\{\frac{r}{2|f_{\mathtt{i}|n,x}(\hat{c}_1)|}, \frac{r}{2|f_{\mathtt{i}|n,x}(\hat{c}_2)|}\right\} \geq \frac{r}{2\max_{\mathbf{b}\in[0,1]^2}|f_{\mathtt{i}|n,x}(\mathbf{b})|}$$

$$\geq \frac{M^{-1}r}{2\max_{\mathbf{b}\in[0,1]^2} \alpha_1 \left(D_{\mathbf{b}} S_{\mathrm{i}|n}\right)}.$$

Therefore

$$\sum_{\mathrm{j}\in\mathcal{J}} m([\mathrm{j}]) \leq \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^n \mathbf{i})), \frac{Mr}{\min_{\mathbf{b}\in[0,1]^2} \alpha_1 \left(D_{\mathbf{b}} S_{\mathrm{i}|n}\right)}\right)\right)$$

and

$$\sum_{\mathrm{j}\in\mathcal{J}} m([\mathrm{j}]) \geq \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^n \mathbf{i})), \frac{M^{-1}r}{\max_{\mathbf{b}\in[0,1]^2} \alpha_1 \left(D_{\mathbf{b}} S_{\mathrm{i}}\right)}\right)\right).$$

Combining this with (4.3.5) completes the proof.

□

Recall by [21, Theorem 2.8] that as an ergodic measure on a self-conformal set, $\pi(\mu)$ is exact dimensional with exact dimension $\dim \pi(\mu)$. This informs us how

$$\pi(\mu)\left(Q_1\left(\pi(\Pi(\mathbf{i})), \frac{1}{n}\right)\right) \tag{4.3.6}$$

scales for an m-typical point $\mathbf{i} \in \Sigma$, although it does not provide any uniform bounds on the projected measure of this interval. The following lemma guarantees the existence of a set of positive measure on which we can uniformly bound (4.3.6).

Lemma 4.3.7. *Let* $\dim \pi(\mu) = t$. *There exists a set* $G \subseteq \Sigma$ *with measure* $m(G) \geq 1/2$ *such that if* $\varepsilon > 0$, *then for all* n *sufficiently large*

$$\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\mathbf{i})), \frac{1}{n}\right)\right) \leq (t-\varepsilon)\log\left(\frac{1}{n}\right)$$

and

$$\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\mathbf{i})), \frac{1}{n}\right)\right) \geq (t+\varepsilon)\log\left(\frac{1}{n}\right)$$

for all $\mathtt{i} \in G$.

Proof. Define $f_n : \Sigma \to \mathbb{R}$ by

$$f_n(\mathtt{i}) = \frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\mathtt{i})), \frac{1}{n}\right)\right)}{-\log n}.$$

Therefore for m-almost all $\mathtt{i}$

$$\lim_{n\to\infty} f_n(\mathtt{i}) = \lim_{n\to\infty} \frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\mathtt{i})), \frac{1}{n}\right)\right)}{-\log n} = t$$

because $\pi(\mu)$ is exact dimensional. By Egorov's Theorem there exists a Borel measurable set $G \subseteq \Sigma$ with $m(G) \geq 1/2$ such that f_n converges uniformly on G. In particular, for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$t - \varepsilon \leq \frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\mathtt{i})), \frac{1}{n}\right)\right)}{-\log n} \leq t + \varepsilon$$

for all $n \geq N_\varepsilon$ and $\mathtt{i} \in G$. Rearranging this expression yields the desired result. □

Next we show that for m-almost all $\mathtt{i} \in \Sigma$ the sequence of points $\{\sigma^n(\mathtt{i})\}_{n\in\mathbb{N}}$ regularly visits the set G from Lemma 4.3.7, yielding uniform bounds on the projected measure of the intervals that appear in (4.3.3) and (4.3.4) along a subsequence of $n \in \mathbb{N}$.

Lemma 4.3.8. *Let* $\dim \pi(\mu) = t$ *and for each* $\mathtt{i} \in \Sigma$ *let* $(r_n(\mathtt{i}))_{n\in\mathbb{N}}$ *be a positive null sequence such that* $r_n(\mathtt{i}) \to 0$ *uniformly over all* $\mathtt{i} \in \Sigma$. *Then for* m*-almost all* $\mathtt{i} \in \Sigma$ *there exists a sequence* $\{n_k\}_{k\in\mathbb{N}}$ *such that for all* $\varepsilon > 0$

$$\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathtt{i})), r_{n_k}(\mathtt{i})\right)\right) \leq (t-\varepsilon)\log\left(r_{n_k}(\mathtt{i})\right) \tag{4.3.9}$$

and

$$\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathtt{i})), r_{n_k}(\mathtt{i})\right)\right) \geq (t+\varepsilon)\log\left(r_{n_k}(\mathtt{i})\right) \tag{4.3.10}$$

for all sufficiently large $k \in \mathbb{N}$*. Furthermore the sequence* $\{n_k\}_{k\in\mathbb{N}}$ *can be chosen to satisfy*

$$\lim_{k\to\infty}\frac{n_{k+1}}{n_k} = 1.$$

Proof. Let G be the set from the statement of Lemma 4.3.7 with characteristic function $\mathbf{1}_G$, which is easily seen to be in $L^1(\Sigma)$. We can now apply the Birkhoff Ergodic Theorem to obtain that for m-almost all $\mathtt{i} \in \Sigma$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\mathbf{1}_G(\sigma^j\mathtt{i}) = \int \mathbf{1}_G dm = m(G) \geq 1/2.$$

This gives that for m-almost all $\mathtt{i} \in \Sigma$, $\sigma^j\mathtt{i} \in G$ with frequency greater than or equal to 1/2. For each $\mathtt{i} \in \Sigma$ which satisfies this let $\{n_k\}_{k\in\mathbb{N}}$ be the sequence for which $\sigma^{n_k}\mathtt{i} \in G$ for all $k \in \mathbb{N}$. Then by Lemma 4.3.7 for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathtt{i})), \frac{1}{n}\right)\right) \leq \left(t-\frac{\varepsilon}{2}\right)\log\left(\frac{1}{n}\right)$$

and

$$\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathtt{i})), \frac{1}{n}\right)\right) \geq \left(t+\frac{\varepsilon}{2}\right)\log\left(\frac{1}{n}\right)$$

for $n \geq N_\varepsilon$ and all $k \in \mathbb{N}$. Since $r_n(\mathtt{i}) \to 0$ uniformly over all $\mathtt{i} \in \Sigma$, we can choose $M_\varepsilon \in \mathbb{N}$ such that $r_n(\mathtt{i}) \leq \frac{1}{N_\varepsilon}$ for all $n \geq M_\varepsilon$. In particular for all $n_k \geq M_\varepsilon$ and m-almost all $\mathtt{i} \in \Sigma$ there exists $l \geq N_\varepsilon$ such that

$$\frac{1}{l+1} \leq r_{n_k}(\mathtt{i}) \leq \frac{1}{l},$$

which gives

$$\frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathfrak{i})), \frac{1}{l}\right)\right)}{\log\left(\frac{1}{l}\right) + \log\left(\frac{l}{l+1}\right)} \leq \frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathfrak{i})), r_{n_k}(\mathfrak{i})\right)\right)}{\log r_{n_k}(\mathfrak{i})}$$

and

$$\frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathfrak{i})), r_{n_k}(\mathfrak{i})\right)\right)}{\log r_{n_k}(\mathfrak{i})} \leq \frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathfrak{i})), \frac{1}{l+1}\right)\right)}{\log\left(\frac{1}{l+1}\right) + \log\left(\frac{l+1}{l}\right)}.$$

Hence there exists $N'_\varepsilon \geq M_\varepsilon$ such that for m-almost all all $\mathfrak{i}$ and all $n_k \geq N'_\varepsilon$,

$$\left|\frac{\log \pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathfrak{i})), r_{n_k}(\mathfrak{i})\right)\right)}{\log r_{n_k}(\mathfrak{i})} - t\right| \leq \varepsilon,$$

giving the first part of the result.

It remains to show that $\lim_{k\to\infty} n_{k+1}/n_k = 1$. Let $\mathfrak{i}$ belong to the set of full m-measure for which $\sigma^j\mathfrak{i} \in G$ with frequency at least 1/2 and write

$$S_{n_k} = \sum_{j=0}^{n_k-1} \mathbf{1}_G(\sigma^j\mathfrak{i}).$$

By Birkhoff's Ergodic Theorem $\lim_{k\to\infty} S_{n_k}/n_k = m(G) \geq 1/2$ and clearly $S_{n_{k+1}} = S_{n_k} + 1$. This implies that

$$\frac{n_{k+1}}{n_k} = \frac{S_{n_{k+1}}}{n_k}\frac{n_{k+1}}{S_{n_{k+1}}} = \left(\frac{S_{n_k}+1}{n_k}\right)\frac{n_{k+1}}{S_{n_{k+1}}} \to \frac{m(G)}{m(G)} = 1$$

as $k \to \infty$, completing the result. □

The RSSC gives us control over the distance between "level n" cylinders, as described in the following lemma.

Lemma 4.3.11. *There exists $\theta > 0$ such that for any $\mathtt{i}, \mathtt{l} \in \Sigma$ with $\mathtt{i}|n \neq \mathtt{l}|n$,*

$$\inf_{\mathbf{a},\mathbf{b}\in[0,1]^2} d(S_{\mathtt{i}|n}(\mathbf{a}), S_{\mathtt{l}|_n}(\mathbf{b})) \geq \theta\alpha_2(D_{\Pi(\sigma^{n-1}\mathtt{i})}S_{\mathtt{i}|_{n-1}}),$$

where d denotes the standard Euclidean metric.

Proof. We begin by showing that for any $a_1, b_1, b_2 \in [0,1]$ with $a_1 \neq b_1$ and any $\mathtt{i} \in \mathcal{I}^*$,

$$\frac{|g_{\mathtt{i}}(b_1, b_2) - g_{\mathtt{i}}(a_1, b_2)|}{|f_{\mathtt{i}}(b_1) - f_{\mathtt{i}}(a_1)|} \leq C \tag{4.3.12}$$

where C is the constant from Lemma 3.4.11. To see this, notice that by the mean value theorem there exist $c_1, c_2 \in (a_1, b_1)$ such that

$$\frac{|g_{\mathtt{i}}(b_1, b_2) - g_{\mathtt{i}}(a_1, b_2)|}{|f_{\mathtt{i}}(b_1) - f_{\mathtt{i}}(a_1)|} = \frac{|g_{\mathtt{i},x}(c_1, b_2)||b_1 - a_1|}{|f_{\mathtt{i},x}(c_2)||b_1 - a_1|} = \frac{|g_{\mathtt{i},x}(c_1, b_2)|}{|f_{\mathtt{i},x}(c_2)|} \leq C,$$

where the final inequality follows by (3.4.12).

Now, let $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in [0,1]^2$. Define $\mathbf{c} = (b_1, a_2)$. We will now show that

$$d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{b})) \gtrsim d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{c})) + d(S_{\mathtt{i}}(\mathbf{c}), S_{\mathtt{i}}(\mathbf{b})), \tag{4.3.13}$$

where the implied constant is independent of $\mathtt{i} \in \mathcal{I}^*$, $\mathbf{a}$, and $\mathbf{b}$. To see this, we let $\gamma = |f_{\mathtt{i}}(a_1) - f_{\mathtt{i}}(b_1)|$, $\varepsilon = |g_{\mathtt{i}}(\mathbf{a}) - g_{\mathtt{i}}(\mathbf{b})|$ and $\eta = |g_{\mathtt{i}}(\mathbf{a}) - g_{\mathtt{i}}(\mathbf{c})|$. Note that $d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{b})) = \sqrt{\gamma^2 + \varepsilon^2}$, $d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{c})) = \sqrt{\gamma^2 + \eta^2}$. This is displayed visually in Figure 4.1.

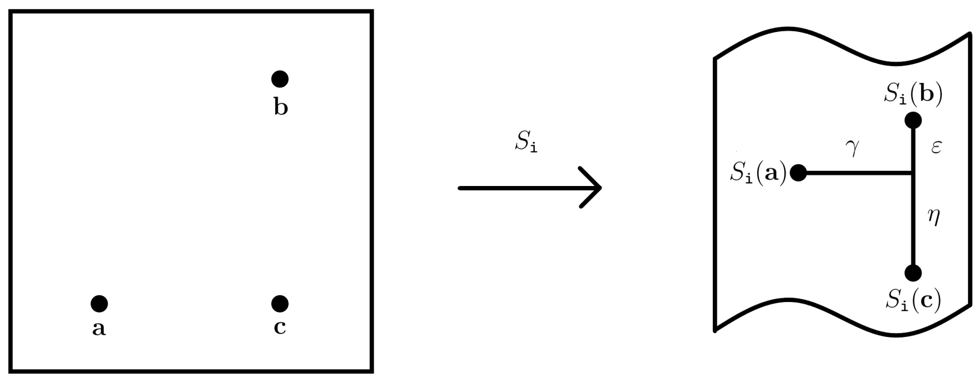

Figure 4.1: The images of the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ under $S_{\mathtt{i}}$ and the distances γ, ϵ and η.

There are now three possibilities: (i) $d(S_{\mathtt{i}}(\mathbf{c}), S_{\mathtt{i}}(\mathbf{b})) = \eta + \varepsilon$, (ii) $\eta > \varepsilon$ and $d(S_{\mathtt{i}}(\mathbf{c}), S_{\mathtt{i}}(\mathbf{b})) = \eta - \varepsilon$ or (iii) $\varepsilon > \eta$ and $d(S_{\mathtt{i}}(\mathbf{c}), S_{\mathtt{i}}(\mathbf{b})) = \varepsilon - \eta$. Hence

$$\frac{d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{c})) + d(S_{\mathtt{i}}(\mathbf{c}), S_{\mathtt{i}}(\mathbf{b}))}{d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{b}))} = \frac{\sqrt{\gamma^2 + \eta^2} + d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{c}))}{\sqrt{\gamma^2 + \varepsilon^2}}.$$

In cases (i) and (iii) we can use (4.3.12) to bound $\eta \lesssim \gamma$, yielding that

$$\frac{\sqrt{\gamma^2 + \eta^2} + d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{c}))}{\sqrt{\gamma^2 + \varepsilon^2}} \lesssim \frac{\gamma + \varepsilon}{\sqrt{\gamma^2 + \varepsilon^2}} \lesssim 1,$$

whereas in case (ii) we can use $\eta \lesssim \gamma$ to deduce that

$$\frac{\sqrt{\gamma^2 + \eta^2} + d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{c}))}{\sqrt{\gamma^2 + \varepsilon^2}} \lesssim \frac{\gamma}{\gamma} = 1.$$

This completes the proof of (4.3.13).

Now, notice that by the mean value theorem there exists $\mathbf{c}_1 \in [0,1]^2$ such that

$$d(S_{\mathtt{i}}(\mathbf{a}), S_{\mathtt{i}}(\mathbf{c}))^2 = f_{\mathtt{i},x}(\mathbf{c}_1)^2 |a_1 - b_1|^2 + g_{\mathtt{i},x}(\mathbf{c}_1)^2 |a_1 - b_1|^2$$

$$\geq \quad d(\mathbf{a},\mathbf{c})^2 f_{\mathrm{i},x}(\mathbf{c}_1)^2 \geq d(\mathbf{a},\mathbf{c})^2 \sup_{\mathbf{c}_2\in[0,1]^2} g_{\mathrm{i},y}(\mathbf{c}_2)^2$$

by (3.4.2). Similarly one can check that

$$d(S_{\mathrm{i}}(\mathbf{b}),S_{\mathrm{i}}(\mathbf{c}))^2 \geq d(\mathbf{b},\mathbf{c})^2 \inf_{\mathbf{c}_2\in[0,1]^2} g_{\mathrm{i},y}(\mathbf{c}_2)^2.$$

Therefore

$$\begin{aligned}
d(S_{\mathrm{i}}(\mathbf{a}),S_{\mathrm{i}}(\mathbf{b})) &\gtrsim d(S_{\mathrm{i}}(\mathbf{a}),S_{\mathrm{i}}(\mathbf{c}))+d(S_{\mathrm{i}}(\mathbf{c}),S_{\mathrm{i}}(\mathbf{b})) \\
&\gtrsim (d(\mathbf{a},\mathbf{c})+d(\mathbf{b},\mathbf{c})) \inf_{\mathbf{c}_2\in[0,1]^2} |g_{\mathrm{i},y}(\mathbf{c}_2)| \\
&\geq d(\mathbf{a},\mathbf{b}) \inf_{\mathbf{c}_2\in[0,1]^2} |g_{\mathrm{i},y}(\mathbf{c}_2)| \\
&\gtrsim d(\mathbf{a},\mathbf{b}) \sup_{\mathbf{c}_3\in[0,1]^2} \alpha_2(D_{\mathbf{c}_3}S_{\mathrm{i}}), \qquad (4.3.14)
\end{aligned}$$

where the first inequality follows by (4.3.13) and the final one by Lemma 3.4.13.

Finally, by the RSSC, there exists $\delta>0$ such that

$$\min_{i\neq j\in\mathcal{I}} \min_{x\in S_i(F)} \min_{y\in S_j(F)} d(x,y)\geq\delta. \qquad (4.3.15)$$

Let $\mathfrak{i}=(i_1,i_2,\ldots),\mathfrak{l}=(l_1,l_2,\ldots)\in\Sigma$ with $\mathfrak{i}|n\neq\mathfrak{l}|n$. In particular there exists $0\leq m\leq n-1$ such that $\mathfrak{i}|m=\mathfrak{l}|m$ and $i_{m+1}\neq l_{m+1}$. We write $\mathfrak{i}|n=\mathfrak{i}|m\mathfrak{j}$ and $\mathfrak{l}|n=\mathfrak{i}|m\mathfrak{r}$. Then for all $\mathbf{a},\mathbf{b}\in[0,1]$

$$\begin{aligned}
d(S_{\mathrm{i}|n}(\mathbf{a}),S_{\mathfrak{l}|n}(\mathbf{b})) &= d(S_{\mathrm{i}|m}(S_{\mathfrak{j}}(\mathbf{a})),S_{\mathrm{i}|m}(S_{\mathfrak{r}}(\mathbf{b}))) \\
&\gtrsim d(S_{\mathfrak{j}}(\mathbf{a}),S_{\mathfrak{r}}(\mathbf{b})) \sup_{\mathbf{c}_3\in[0,1]^2} \alpha_2(D_{\mathbf{c}_3}S_{\mathrm{i}|m}) \\
&\gtrsim \alpha_2(D_{\Pi(\sigma^{n-1}\mathrm{i})}S_{\mathrm{i}|n-1}),
\end{aligned}$$

where the second inequality follows by (4.3.14) and the final inequality follows by

(4.3.15) (since $\mathfrak{j}$ and $\mathfrak{r}$ begin with different digits) and (3.4.15). $\square$

We are now in a position to be able to prove Theorem 4.2.5, our main result. We do so by establishing both the corresponding lower and upper bounds for the local dimension of μ at $\Pi(\mathfrak{i})$ for $\mathfrak{i} \in \Sigma$ belonging to a set of full m-measure. It is worth noting that of the two bounds only the lower one requires Lemma 4.3.11 and as such this is the only one of the two that requires the RSSC.

Proof of Theorem 4.2.5. Let $\mathfrak{i} \in \Sigma$ belong to the set of full m-measure for which (1.7.7), (4.2.3) and (4.2.4) hold. As in (3.4.3) let

$$\eta := \sup_{i \in \mathcal{I}, \mathbf{a}, \mathbf{b} \in [0,1]^2} \left\{ \frac{|g_{i,y}(\mathbf{a})|}{|f_{i,x}(\mathbf{b})|} \right\}$$

and recall that $\eta < 1$ by the domination condition (Definition 3.4.1). Using Lemma 3.4.13, applying the chain rule to $g_{\mathfrak{i}|n,y}(\Pi(\sigma^n \mathfrak{i}))$ and $f_{\mathfrak{i}|n,x}(\mathbf{b})$ for each $n \in \mathbb{N}$ and pairing off appropriate terms we get

$$\frac{\alpha_2 \left(D_{\Pi(\sigma^n \mathfrak{i})} S_{\mathfrak{i}|n} \right)}{\min_{\mathbf{b} \in [0,1]^2} \alpha_1 \left(D_{\mathbf{b}} S_{\mathfrak{i}|n} \right)} \leq \frac{M^2 |g_{\mathfrak{i}|n,y}(\Pi(\sigma^n \mathfrak{i}))|}{\min_{\mathbf{b} \in [0,1]^2} |f_{\mathfrak{i}|n,x}(\mathbf{b})|} \leq M^2 \eta^n \to 0.$$

Similarly,

$$\frac{\alpha_2 \left(D_{\Pi(\sigma^n \mathfrak{i})} S_{\mathfrak{i}|n} \right)}{\max_{\mathbf{b} \in [0,1]^2} \alpha_1 \left(D_{\mathbf{b}} S_{\mathfrak{i}|n} \right)} \leq M^2 \eta^n \to 0.$$

Define the sequences

$$r_n(\mathfrak{i}) = \frac{M \theta \alpha_2 \left(D_{\Pi(\sigma^n \mathfrak{i})} S_{\mathfrak{i}|n} \right)}{\min_{\mathbf{b} \in [0,1]^2} \alpha_1 \left(D_{\mathbf{b}} S_{\mathfrak{i}|n} \right)} \quad \text{and} \quad r'_n(\mathfrak{i}) = \frac{M^{-1} \alpha_2 \left(D_{\Pi(\sigma^n \mathfrak{i})} S_{\mathfrak{i}|n} \right)}{\max_{\mathbf{b} \in [0,1]^2} \alpha_1 \left(D_{\mathbf{b}} S_{\mathfrak{i}|n} \right)}, \tag{4.3.16}$$

and observe that both $r_n(\mathfrak{i})$ and $r'_n(\mathfrak{i})$ converge to 0 uniformly over all $\mathfrak{i} \in \Sigma$. Hence we can also assume that $\mathfrak{i} \in \Sigma$ belongs to the set of full measure which satisfies (4.3.9) and

(4.3.10) for the sequences $r_n(\mathbf{i})$ and $r'_n(\mathbf{i})$.

For a given $\mathbf{a} = (a_1, a_2) \in [0,1]^2$ and $r > 0$ write

$$Q_2(\mathbf{a}, r) = \left(a_1 - \frac{r}{2}, a_1 + \frac{r}{2}\right) \times \left(a_2 - \frac{r}{2}, a_2 + \frac{r}{2}\right).$$

Write $\mathbf{x} = \Pi(\mathbf{i})$, let $n \in \mathbb{N}$ and consider the square $Q_2\left(\mathbf{x}, \theta\alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right)\right)$. By Lemma 4.3.11 $Q_2\left(\mathbf{x}, \theta\alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right)\right)$ intersects only the cylinder $S_{\mathrm{i}|n}([0,1]^2)$, therefore it is easy to see that

$$Q_2\left(\mathbf{x}, \theta\alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right)\right) \cap F \subseteq B_n\left(\mathbf{x}, \theta\alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right)\right).$$

Hence by Lemma 4.3.2,

$$\begin{aligned}
&\mu\left(Q_2\left(\mathbf{x}, \theta\alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right)\right)\right) \\
&\leq Lm([\mathbf{i}|n])\pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^n \mathbf{i})), \frac{M\theta\alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right)}{\min_{\mathbf{b}\in[0,1]^2} \alpha_1\left(D_{\mathbf{b}} S_{\mathrm{i}|n}\right)}\right)\right).
\end{aligned} \tag{4.3.17}$$

We now turn our attention to $\alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right)$. Applying the chain rule gives

$$\begin{aligned}
\alpha_2\left(D_{\Pi(\sigma^{n+1} \mathrm{i})} S_{\mathrm{i}|n+1}\right) &= \alpha_2\left(D_{S_{i_{n+1}}\Pi(\sigma^{n+1} \mathrm{i})} S_{\mathrm{i}|n} D_{\Pi(\sigma^{n+1} \mathrm{i})} S_{i_{n+1}}\right) \\
&\leq \alpha_2\left(D_{S_{i_{n+1}}\Pi(\sigma^{n+1} \mathrm{i})} S_{\mathrm{i}|n}\right) \alpha_1\left(D_{\Pi(\sigma^{n+1} \mathrm{i})} S_{i_{n+1}}\right) \\
&< \alpha_2\left(D_{S_{i_{n+1}}\Pi(\sigma^{n+1} \mathrm{i})} S_{\mathrm{i}|n}\right) \\
&= \alpha_2\left(D_{\Pi(\sigma^n \mathrm{i})} S_{\mathrm{i}|n}\right),
\end{aligned} \tag{4.3.18}$$

where in the first inequality we have used that $\alpha_2(AB) \leq \alpha_2(A)\alpha_1(B)$ for 2×2 matrices A and B. Consider the subsequence $(n_k)_{k\in\mathbb{N}}$ guaranteed by applying Lemma 4.3.8 to the sequence $r_n(\mathbf{i})$. Then (4.3.18) implies that the null subsequence $\left(\theta\alpha_2\left(D_{\Pi(\sigma^{n_k} \mathrm{i})} S_{\mathrm{i}|n_k}\right)\right)_{k\in\mathbb{N}}$

is strictly decreasing. Hence for any $r > 0$ sufficiently small we can choose $k \in \mathbb{N}$ sufficiently large so that

$$\theta\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|n_{k+1}}\right) \leq r < \theta\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right). \tag{4.3.19}$$

Let $t = \dim \pi(\mu)$ and $\varepsilon > 0$. Let $r > 0$ be sufficiently small so that $k \in \mathbb{N}$ that satisfies (4.3.19) is sufficiently large that (4.3.9) holds for ε. Then, using (4.3.1), (4.3.17) and (4.3.9) we get

$$\begin{aligned}
&\frac{\log\mu\left(Q_2(\mathbf{x}, r)\right)}{\log r}\\
&\geq \frac{\log\mu\left(Q_2\left(\mathbf{x}, \theta\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)\right)\right)}{\log\left(\theta\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)\right)}\\
&\geq \frac{\log Lm([\mathbf{i}|n_k]) + \log\pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_k}\mathbf{i})), \frac{M\theta\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)}{\min_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|n_k}\right)}\right)\right)}{\log\left(\theta\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)\right)}\\
&\geq \frac{\log Lm([\mathbf{i}|n_k]) + (t-\varepsilon)\log\left(\frac{M\theta\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)}{\min_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|n_k}\right)}\right)}{\log\left(\theta\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)\right)}\\
&= \frac{-\frac{1}{n_k}\log L - \frac{1}{n_k}\log m([\mathbf{i}|n_k]) - \frac{(t-\varepsilon)}{n_k}\log\left(\frac{\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)}{\min_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|n_k}\right)}\right) - \frac{(t-\varepsilon)}{n_k}\log M\theta}{-\frac{1}{n_k}\log\theta - \frac{1}{n_{k+1}}\frac{n_{k+1}}{n_k}\log\left(\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)\right)}\\
&\to \frac{h(\mu) + (t-\varepsilon)(\chi_2(\mu) - \chi_1(\mu))}{\chi_2(\mu)}
\end{aligned}$$

as $r \to 0$ (so $k \to \infty$). Since $\varepsilon > 0$ was arbitrary, the lower bound is complete.

We now establish the corresponding upper bound. Let $n \in \mathbb{N}$. For some $a, b \in [0, 1]$ with the property that

$$|f_{\mathtt{i}|n}(b) - f_{\mathtt{i}|n}(a)| \leq \alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right), \tag{4.3.20}$$

we can write

$$\sup_{(a_1,a_2),(b_1,b_2)\in B_n(\mathbf{x},\alpha_2(D_{\Pi(\sigma^n(\mathtt{i}))}S_{\mathtt{i}|n}))} |a_2 - b_2| = |g_{\mathtt{i}|n}(b,1) - g_{\mathtt{i}|n}(a,0)|.$$

Note that

$$\begin{aligned}|g_{\mathtt{i}|n}(b,1) - g_{\mathtt{i}|n}(a,0)| &\le |g_{\mathtt{i}|n}(b,1) - g_{\mathtt{i}|n}(a,1)| + |g_{\mathtt{i}|n}(a,1) - g_{\mathtt{i}|n}(a,0)| \\ &\le C|f_{\mathtt{i}|n}(b) - f_{\mathtt{i}|n}(a)| + |g_{\mathtt{i}|n,y}(\mathbf{c})|\end{aligned}$$

for some $\mathbf{c} \in [0,1]^2$ where we have used (4.3.12) and the mean value theorem. Lemmas 3.4.9 and 3.4.13 then allow us to bound $|g_{\mathtt{i}|n,y}(\mathbf{c})|$, giving

$$|g_{\mathtt{i}|n,y}(\mathbf{c})| \le MR\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right).$$

Thus it follows from (4.3.20) that

$$\begin{aligned}|g_{\mathtt{i}|n}(b,1) - g_{\mathtt{i}|n}(a,0)| &\le C\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right) + MR\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right) \\ &= (MR + C)\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right).\end{aligned}$$

It is now easy to see that

$$B_n\left(\mathbf{x}, \alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right)\right) \cap F \subseteq Q_2\left(\mathbf{x}, 2(MR+C)\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right)\right) \cap F.$$

Thus Lemma 4.3.2 implies

$$\begin{aligned}&\mu\left(Q_2\left(\mathbf{x}, 2(MR+C)\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right)\right)\right) \\ &\ge L^{-1}m([\mathtt{i}|n])\pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^n\mathbf{i})), \frac{M^{-1}\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right)}{\max_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|n}\right)}\right)\right). \qquad (4.3.21)\end{aligned}$$

Consider the subsequence $(n_k)_{k\in\mathbb{N}}$ guaranteed by applying Lemma 4.3.8 to $r'_n(\mathtt{i})$, which was defined in (4.3.16). Since $\left(\alpha_2\left(D_{\Pi(\sigma^n\mathtt{i})}S_{\mathtt{i}|n}\right)\right)_{n\in\mathbb{N}}$ is strictly decreasing and null, for any $r>0$ sufficiently small we can choose $k\in\mathbb{N}$ sufficiently large so that

$$2(MR+C)\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)\le r<2(MR+C)\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right). \qquad (4.3.22)$$

Let $\varepsilon>0$. Let $r>0$ be sufficiently small so that $k\in\mathbb{N}$ that satisfies (4.3.22) is sufficiently large that (4.3.10) holds for ε. Therefore by using (4.3.1), (4.3.21) and (4.3.10) we get

$$\begin{aligned}
&\frac{\log\mu\left(Q_2(\mathbf{x},r)\right)}{\log r}\\
&\le\frac{\log\mu\left(Q_2\left(\mathbf{x},2(MR+C)\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)\right)\right)}{\log\left(2(MR+C)\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)\right)}\\
&\le\frac{\log L^{-1}m([\mathtt{i}|_{n_{k+1}}])+\log\pi(\mu)\left(Q_1\left(\pi(\Pi(\sigma^{n_{k+1}}\mathtt{i})),\frac{M^{-1}\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)}{\max_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|_{n_{k+1}}}\right)}\right)\right)}{\log\left(2(MR+C)\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)\right)}\\
&\le\frac{\log L^{-1}m([\mathtt{i}|_{n_{k+1}}])+(t+\varepsilon)\log\left(\frac{M^{-1}\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)}{\max_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|_{n_{k+1}}}\right)}\right)}{\log\left(2(MR+C)\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)\right)}\\
&=\frac{-\frac{1}{n_{k+1}}\log L^{-1}-\frac{1}{n_{k+1}}\log m([\mathtt{i}|_{n_{k+1}}])-\frac{(t+\varepsilon)}{n_{k+1}}\log\left(\frac{\alpha_2\left(D_{\Pi(\sigma^{n_{k+1}}\mathtt{i})}S_{\mathtt{i}|_{n_{k+1}}}\right)}{\max_{\mathbf{b}\in[0,1]^2}\alpha_1\left(D_{\mathbf{b}}S_{\mathtt{i}|_{n_{k+1}}}\right)}\right)-\frac{(t+\varepsilon)}{n_{k+1}}\log M^{-1}}{-\frac{1}{n_{k+1}}\log 2(MR+C)-\frac{1}{n_k}\frac{n_k}{n_{k+1}}\log\left(\alpha_2\left(D_{\Pi(\sigma^{n_k}\mathtt{i})}S_{\mathtt{i}|n_k}\right)\right)}\\
&\to\frac{h(\mu)+(t+\varepsilon)(\chi_2(\mu)-\chi_1(\mu))}{\chi_2(\mu)}
\end{aligned}$$

as $r\to 0$ (so $k\to\infty$). Since $\varepsilon>0$ was arbitrary, the upper bound follows. □

List of notation

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